PLANE GEOMETRY

S. PADER AND RHOADS

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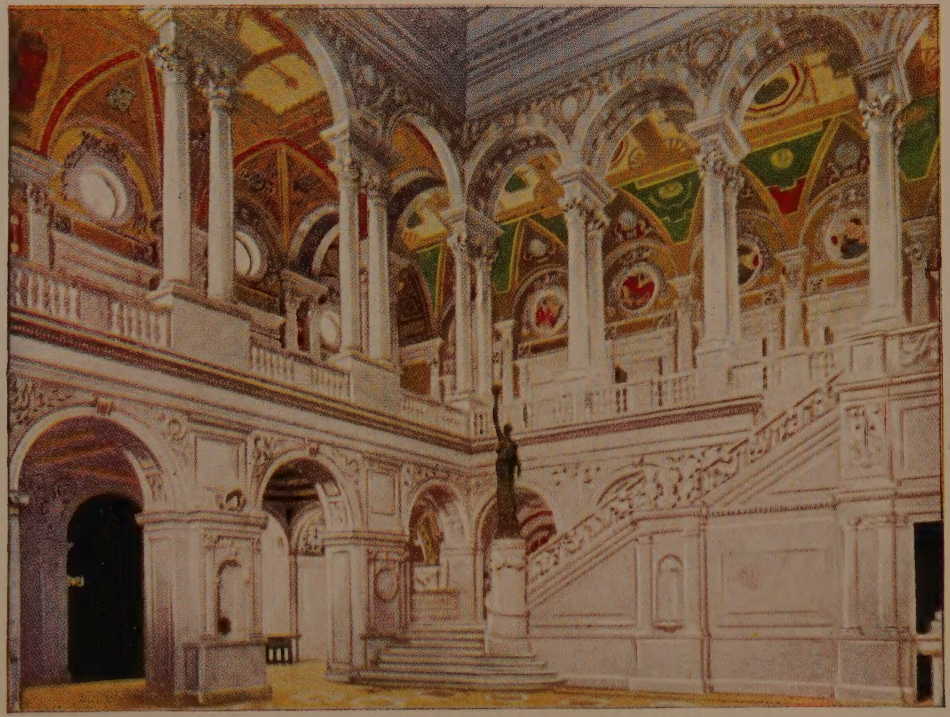

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PLANE GEOMETRY

A MODERN

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BY

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Geometry

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PREFACE

This book aims to provide an interesting and thorough course in plane geometry in accord with the more recent advancements in methods of teaching the subject. It is believed that the treatment as a whole will commend itself to teachers because of its simplicity, its clearness, and its thoroughness.

The Introduction is so designed that at the beginning the pupil may easily obtain by himself an idea of the fundamental concepts and terminology of geometry; then he is led gradually into the formal proof by the use of simple one-step exercises which involve principally the use of angles. Oral exercises always follow a group of related definitions to clinch their meaning and to illustrate their use.

The propositions have been appropriately lettered (**C, c, N,** and **n**) to indicate the requirements of the College Entrance Examination Board and the recommendations of the National Committee. The requirements of other leading syllabi have also been followed. (For Key to the Lettering of Propositions, cf. § 57.)

The statements of many theorems have been simplified and condensed by placing the subject (or hypothesis) first and the predicate (or conclusion) last. This arrangement is simple, logical, and emphatic. (cf. §§ 65, 120, 211, 301, 305, etc.)

The proofs are simple and rigorous. Simplicity of proof has been secured in three ways: first, by choosing a sequence which has eliminated a number of indirect proofs and any necessity for the commensurable and incommensurable proofs of certain theorems; second, by a new order of

sequence in a few instances; third, by changes within the proofs themselves. Rigor of thought has been obtained from beginning to end by supporting each statement of the argument by a previously established fact. Assumptions are explicitly stated and are not implied. Definitions are precise and concise.

The difficulties of the pupil have been anticipated and reduced by the use of the following helpful devices:

a) The *Hint* which precedes a proof suggests the "key" to its solution.

b) The application of special methods, such as the *Corollary Method*, the *Coincidence Method*, the *Substitution Method*, and others, classifies and systematizes the demonstrations.

c) The theorems which are especially important in the subsequent development of the text are named *Tools*. (A more complete explanation of Tools is given in § 73.) The Tools are emphasized by being placed in separate sections and are suggested for use in solving groups of exercises.

d) Numerous Oral Exercises compel thought on the part of the pupil. They clinch the meaning of the definitions, principles, or theorems which they follow and in other cases prepare the student for more difficult written exercises or for the next theorem.

e) Development exercises are used to introduce new kinds of work. (cf. exercises in connection with §§ 45, 87, 88, 175, 312, etc.)

f) Specific directions are given for proving certain families of theorems. (cf. §§ 88, 310, 311, 312, 377, 403, 432, etc.)

g) Summaries of Families of Theorems are added for continuity, emphasis, and reference. (cf. §§ 124–130, 231, 259, 302, 383.)

Inequalities, *Formal* Constructions, and Loci are deferred to the end of the course. This arrangement relieves the beginning of many troublesome difficulties and focuses the attention of the pupil upon one kind of work at a time; it also permits the teacher to give these parts of geometry whatever emphasis may seem desirable. (See also page ix.) In this text, the analysis of attack in Construction Problems (given in the *Hints*) serves to review past work. There is also a continuity of subject matter of each of these three topics by this arrangement. The text, however, is flexible and teachers who wish to introduce these subjects in the usual places may do so without disturbing any other material. (For such sequence refer to the Suggestions to Teachers.)

The written exercises are classified under appropriate headings to follow each important theorem or group of theorems. Only those exercises are included which the average pupil can do. Additional exercises may be taken from the lists of Supplementary Exercises at the close of the chapters. All these exercises have been carefully graded and are numerous enough for a wide selection.

Among other special features are: The Summaries of Theorems at the close of the chapters; the wide variety of unusual and interesting illustrations; the Summary of Formulas; the Miscellaneous Exercises at the close of the chapters and the General Miscellaneous Exercises in the Supplement; the list of practical Applications, which are within the understanding of the average girl or boy; the Recreations, which are amusing and illustrative; the Word List for spelling and vocabulary purposes; the list of terms with the "dictionary form of definition" for reviews and ready reference; the Brief Historical Sketch of Geometry; the Review Oral Exercises at the end of each chapter for rapid diagnosis and remedial work; the reasonable and rigorous treatment of limits by the use of substitution; the

treatment of indirect proofs and exercises for the same; the review of those tools of algebra needed in geometry; the Notion of Functionality as developed by certain exercises; the treatment of trigonometric ratios as a part of geometry; and, the examples of the New Type of Geometry Examination based upon the work of the best authorities.

Suggestions to Teachers

The subject matter of this book is thoroughly organized and classified. This refinement will aid the teacher in the presentation of material because it is more orderly and continuous and because the pupil will be able to secure rapid and satisfying results from study outside the classroom. Teachers with experience in teaching geometry will find that the order of topics is flexible enough to fit in with their accustomed course, and that many helpful aids are included which they have wished for in a text. For the teacher with little experience in teaching geometry, we will outline briefly the purpose and use of parts of our text and suggest some general guiding principles for conducting a course.

In order to meet the many different and sometimes conflicting needs in various sections of the country, *this text includes more material than is given ordinarily in any one course.* One of the first problems then, which confronts a teacher, is to select proper and sufficient material for use. The nature of the requirements to be met, the length of time to be given to the course, the number and quality of the pupils, etc., will largely determine the subject matter and the degree of intensity of the work. To aid in the choice of propositions, we have marked with **N** and **n** those which are needed for the development of the content of mathematics as defined by the National Committee and with **C** and **c** those which are required for examination purposes by the College Entrance Examination Board. To aid

in the selection of exercises, we have arranged them under appropriately named headings. All the Oral Exercises should be taken, as they are of great importance in helping to make clear the meaning of definitions, theorems, and developmental work and to point out the significance of newly learned facts. The named groups of exercises which follow the theorems are numerous enough to give sufficient drill to the average class and many are simple enough for *sight work*. The related Supplementary Exercises found at the close of the chapters may be used for additional drill, for reviews, and for extra exercises to be given to rapid pupils. The Applications, arranged by chapters and found in the Supplement, may be used to show how certain facts learned may be applied to a variety of problems and at the same time sustain the interest of the pupils.

The sets of Review Oral Exercises at the close of the chapters were constructed for rapid diagnosis. The replies given by the pupils should be brief and to the point. Long, time-consuming discussions should be avoided. Weaknesses discovered by the use of these questions might be overcome by proper remedial work.

To aid teachers whose time for the course is limited, or for those who wish to know just what comprises a minimum course (to which they may make their own additions), we include here a carefully planned outline of

A Suggested Minimum Course

	TO INCLUDE	TO OMIT
Introduction	§§ 20–39 §§ 42–56	All up to § 20 §§ 40 and 41
Chap. One	§§ 57–123 §§ 131–153 §§ 158–174 §§ 176–181	§§ 124–130 §§ 154–157 § 175 All after §181

A Suggested Minimum Course (*Continued*)

	TO INCLUDE	TO OMIT
Chap. Two	§§ 182–230 § 232	§ 231 All after § 232
Chap. Three	§§ 233–287	§ 288 to end
Chap. Four	§§ 289–324	§ 325 to end
Chap. Five	§§ 334–352 §§ 354–365	§ 353 § 366 to end
Chap. Six	§§ 370–435	§ 436 to end
Supplement	Principles of Geometry	All the remainder

Note: The omissions above include the Supplementary Exercises, Summaries, Review Oral Exercises, and Miscellaneous Exercises of each of the chapters. With classes which have had sufficient preliminary geometric training, work might begin with Section 42.

Following what we believe to be present-day tendencies in the teaching of mathematics, as evidenced by the Report of the National Committee and elsewhere, we have placed the work of Inequalities, *Formal* Construction problems, and Loci in a separate chapter at the close of the course. However, for those who may wish to use this material throughout the chapters, we suggest the following:

Optional Sequence

Inequalities, §§ 370–383, at the end of Chapter One.
Inequalities, §§ 384, 385, at the end of Chapter Two.
Inequalities, §§ 386–388, after § 324, Chapter Four.

Constructions, §§ 389–413, at the end of Chapter Two.
Constructions, §§ 414–424, at the end of Chapter Four.
Constructions, §§ 425–430, at the end of Chapter Five.

Loci, §§ 431–442, at the end of Chapter Two.

The Introduction to Geometry may be developed in about three weeks. It is so arranged that the majority of the exercises may be done *orally* in class. It should be used primarily to make clear the terms and fundamental principles and to familiarize the pupil with the meaning of a proof through the one-step exercises. Although the simplified sequence in the text and the continuous development of subject matter make it possible for a class to progress rapidly once it has gotten under way, we feel that it would be unwise to advance beyond § 262 in Chapter Three, in the first half year. Such arrangement of the course means that a class will complete in the first semester four distinct units of work, namely: the Introduction, Rectilinear Figures, Circles and Their Lines, and Areas. During the second semester, which would begin with Proportions, the remainder of the course may be completed, leaving time enough for a general review with the use of miscellaneous material found in the Supplement.

The fact that formal Construction problems are to be taken up in Chapter Six should not mislead the teacher into believing that drawing exercises should be omitted also until the end of the course. Free-hand drawing or sketching exercises and work with straightedge, compasses, and protractor have been included at appropriate places in the text. Care should be taken by the teacher to see that the pupil does not consume too much time and unnecessary effort in elaborate drawings. All that should be required is a neat representation of the figure under discussion. The most important uses of such work are:

a) to illustrate the definitions of terms;

b) to make clear the meaning of theorems; and,

c) to emphasize especially the hypotheses.

Other special features of the book, such as the Summaries, Miscellaneous Exercises, Recreations, Methods of Proof,

Word List, etc., may be given whatever attention seems advisable to the teacher. We wish to point out the helpfulness of the Substitution Method of Proof as described in § 103. In this method especially, the Conclusion is the guiding sign post in building the proof. It may be proper for us to suggest at this point that although we may speak at times of attacking, tearing apart, or analyzing a theorem or exercise, from the viewpoint of the pupil it is far more important to speak of *building a proof*.

The frame surrounding the Figure, Hypothesis, and Conclusion separates a demonstration into its three natural elements; the statement of the theorem, the derived parts, and the proof. The frame also emphasizes those parts which are needed for Oral Demonstration. Pupils should not be required to draw a similar frame in their written work.

Miscellaneous Suggestions

Begin slowly. This is difficult for some teachers to do. It is better to build a strong foundation for the course than to rush in headlong without preparing the minds of the pupils for what is to come. Correct any loose thought and careless oral expression as soon as they appear. Encourage and guide the slower pupils in their work. This not only aids them, but will bring about a more uniform development of the class. If colored chalk or crayons are used at the beginning (and such a device is often helpful), their use should not be continued beyond the first few propositions. Stress only the important sections of the text. It is not advisable, for example, to emphasize any part of the work previous to § 20. Some material is included in the book solely for the purpose of creating a mental atmosphere.

The earnest, but slow pupil often tries to keep up with the class by resorting to the memorizing of proofs. Stimulate such a pupil by asking him pertinent questions. Ask him to

state in his own words what a theorem means. Have him draw a figure including only the given parts. Fasten his attention to the Conclusion. Give him other Hints, if necessary, than those found in the text to help him recall the *Tools* which may be used to build his proof. Finally ask him to describe just how he will proceed with his work.

Among the numerous suggestions which might be given, we will include some whose use is more or less evident:

a) State a part of a theorem for a pupil and ask him to complete it.

b) Draw a figure on the board and ask a pupil to tell all the facts he knows about it.

c) Have pupils quote the supporting Facts before looking up the section numbered references.

d) Trace the use of a theorem as a Fact.

e) State the definition of a term and ask for the name of the thing defined.

f) Have pupils summarize frequently the theorems which are used to prove certain kinds of conclusions.

g) Select a word from the Word List and ask how it is spelled, what it means, and where it is used.

h) With lists of numerical exercises ask the pupils to state the theorems upon which the calculations depend.

i) Have pupils make lists of original review questions for discussion.

j) Secure a rapid review of the proofs of important theorems by having pupils state only what auxiliary construction lines are drawn, what triangles are proved congruent, equal, or similar, and what particular conclusion follows.

In conclusion may we suggest that the teacher should guard against "riding hobbies" or of becoming "mind-set" in teaching. It is also inadvisable to sacrifice the progress and possible success of a class by trying out novel experi-

ments. One may be well informed about his work by being familiar with the Report of the National Committee, compiled by well-known authorities; by attending the meetings of local, state, or sectional Mathematical Associations; by keeping an active bookshelf supplied with the works of leading investigators; and by subscribing to at least one magazine devoted to the problem of teaching mathematics.

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William W. Strader
Lawrence D. Rhoads

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SYMBOLS AND ABBREVIATIONS

SYMBOLS

$\angle$ angle; $\measuredangle$ angles

$\rightarrow$ approaches, tends to

$\frown$ arc; as $\overset{\frown}{AB}$, read *arc AB*

$\odot$ circle; $\textcircled{s}$ circles

$\cong$ congruent, is congruent to

$=$ equal, equals, equal to, is equal to

$\neq$ is not equal to, unequals

$>$ is greater than

$<$ is less than

$\parallel$ parallel, parallel to, is parallel to

$\parallel s$ parallels

$\square$ parallelogram; $\boxed{s}$ parallelograms

$\perp$ perpendicular, perpendicular to, is perpendicular to

$\perp s$ perpendiculars

π Pi (Greek letter); when referring to a number means 3.1416 or $3\frac{1}{7}$

A', A'' primes; read *A prime, A double prime*

$\sim$ similar, is similar to

A_1, B_2 subscripts; read *A sub-one, B sub-two*

$\triangle$ triangle; $\blacktriangle$ triangles

ABBREVIATIONS

adj.	adjacent
alt.	alternate
Ax.	axiom
circum.	circumference
comp.	complement, complementary
Con.	conclusion
Cons.	construction
Cor.	corollary
corr.	corresponding
cos	cosine
Def.	definition
Ex.	exercise
ext.	exterior
Fig.	figure
hy.	hypotenuse
Hyp.	hypothesis
iden.	identity, identical
int.	interior
isos.	isosceles
Post.	postulate
Prin.	principle
Prob.	problem
Prop.	proposition
quad.	quadrilateral
rect.	rectangle
rt.	right
sin	sine
st.	straight
Sub.	substitution
supp.	supplement, supplementary
tan	tangent

Columbine

Laurel

Iris

Milkweed

Cross Sections of Flowers

Deutzia

Canna

Horse Chestnut

Leaf Arrangement and Veining

Nightshade

Albuca

Pumpkin

Pink

Hibiscus

Pollen Grains (very highly magnified)

A Willow Branch

Cross Sections

An Orange

GEOMETRY IN PLANTS

PLANE GEOMETRY

INTRODUCTION TO GEOMETRY

What You Study in Geometry. In geometry you will study about certain kinds of figures which exist in the works of nature and in the constructions of man. Wherever you look you are almost sure to find some of these figures. Notice the delicate designs of the snowflakes (see p. 270) which have six points or six sides. Do you know that the honey cells of the bee are made in regular hexagonal form and that this gives strength and compactness? The spider's web is a marvelous combination of triangles and other figures (see p. 9). Crystals dug from the earth possess in their cross sections and faces many kinds of polygons. The arrangement of the leaves of plants and the cross sections of the seed vessels of flowers suggest many regularly formed figures. (See opposite page.) Pages of a book, blackboards, doors, and windows are in the shape of rectangles. The face of a clock, the cross sections of pencils and parts of lighting fixtures are circles. The rails of the trolley track, the opposite edges of a ruler, and rows of growing corn appear in parallel lines. Many buildings show pleasing combinations of figures. Notice the wonderful use of rectangles, arcs, and circles in the Entrance Hall of the Congressional Library. (See Frontispiece.) Look at the picture on page 214 and observe how a knowledge of the properties of triangles makes it possible to take an aerial photograph. If you are a boy and try to make a simple plaything like a kite, you will at once realize the necessity for using correct geometric

forms and knowing certain relationships of parts. If you are a girl and hemstitch a handkerchief, you must observe an important relationship of parallel lines. On page 6 notice that the designs shown are composed of a limited number of geometric figures.

Why You Study Geometry. As you study about such figures, their properties, and uses, you will gain not only *knowledge*, but *greater power in correct thinking*. For ages, geometry has stood as the subject best adapted to train girls and boys in the habits of *expressing their thoughts accurately* and of *reasoning correctly*. You will cultivate these habits by forming conclusions about the properties of the simplest and most universally known elements (triangles, rectangles, circles, etc.). Such a training as geometry gives is both essential and desirable for your success in any station of life.

How You Study Geometry. You will observe as you progress in geometry that your work is somewhat like that of a builder. You will lay a foundation for the structure of geometry by defining certain fundamental terms and by accepting some general statements as truths. As the foundation of a building supports the first story and as each story in turn supports the story above it, so in this subject the Introduction to Geometry and the various chapters are each a support to the following work. Any new material which enters into the construction of a building is tested and approved, and in like manner each statement you use in geometry must be proved. The proofs piece together facts which are already known and by this means show new statements to be true. The following exercises may give a hint of the processes employed. They are not mere puzzles to be solved in a "hit or miss fashion"; each statement of the solution should be defended by a reason.

Thought Exercises

1. Which of the following numbers are exactly divisible by 5? 234; 345; 456; 27980; $(125)^4$. Why?

2. The adjoining figure is called a *magic square* of the third order. It is so arranged that the sum of any three numbers in a straight line is 15. Take the numbers 3 to 11 inclusive and try to form a magic square in which these sums are each 21. Why must each sum be 21 and not some other number?

2	9	4
7	5	3
6	1	8

3. Find two ways of showing which of the fractions $\frac{18}{24}$ and $\frac{12}{15}$ is the greater. Be able to defend each step of your work.

4. Explain how you can find the missing figures in the accompanying multiplication.

$$\begin{array}{r} x\,x\,x\,x \\ \times\quad 7 \\ \hline 9\,4\,1\,5 \\ x\,x\,x\,x \\ \hline 6\,3\,2\,1\,5 \end{array}$$

5. Five lights are to be suspended from the ceiling of the schoolroom. Explain what you think might be the best arrangement.

6. Suggest some plan for collecting the written work of your class so that it will be arranged alphabetically.

7. Assume that a cork has been pushed through the neck into a bottle. How would you get the cork out without breaking the bottle?

8. If the Fourth of July falls on Monday, on what day of the week does Columbus Day of the same year fall?

9. If 8 books cost \$20, why do 16 books cost \$40?

10. A picture measures 6″ x 8″ and is surrounded by a frame 1″ wide. Explain by at least *two* methods how to find the area of the frame.

THE ORIGIN OF GEOMETRY

Geometry is supposed to have had its beginning with land surveying by the priests of Egypt. The Nile River periodically overflowed its banks carrying away landmarks in its valley, and by altering its course; increased or decreased the taxable value of the adjoining lands. This necessitated frequent measurements to determine not only the position of property, but also its extent.

Since the early Egyptians were sun worshipers, they erected their temples and principal buildings to face the east

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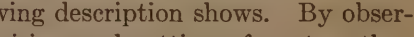
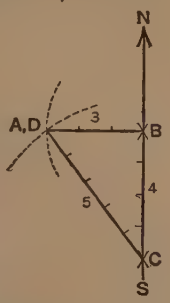
At the left is the Great (Cheops) Pyramid, one of the seven wonders of the ancient world. It is 762 feet square at the base and 484 feet high. The small rectangular mounds between the pyramids are tombs of nobles, commanders, and engineers attached to the courts of the kings who were entombed in the pyramids.

or the rising sun. The pyramids, which were tombs of kings, have four triangular sides which face exactly north, east, south, and west. There were good reasons then why a systematic study of measurements was taken up by the Egyptians.

Enough information has come down to us to say that they were able to locate the four points of the compass in some such manner as the following description shows. By observations based upon the rising and setting of a star, they were able to locate a line running due north and south. To get an east and west line which had to be at right angles to this, professional geometers were employed. These men used a rope $ABCD$ divided by knots or marks at B and C, so that the lengths AB, BC, and CD were in the ratio 3:4:5. The length BC was placed along the *north-south* line and pegged fast at the knots B and C. The piece BA (keeping it stretched all the time) was then rotated about B,



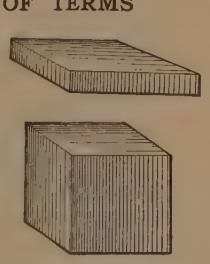
and in like manner the piece CD was rotated about C, until the ends A and D coincided. The line AB then marked an *east-west* position.



The geometry known to the Egyptians corresponds closely to the geometric knowledge you already possess from the study of *Mensuration* in arithmetic. It consisted principally of *rules* or *formulas* for finding *areas* and *volumes*. After a time Greek philosophers and teachers who visited Egypt gave this form of geometry a new development. The Egyptians were principally interested in the practical applications of their rules; the Greeks developed and perfected their proofs. In fact the elementary geometry which we study today is mainly due to the genius of the Greeks. If you should look up the derivation of the words found in the Word List for Plane Geometry, page 361, you would discover that the great majority of geometric terms are from Greek *roots*.

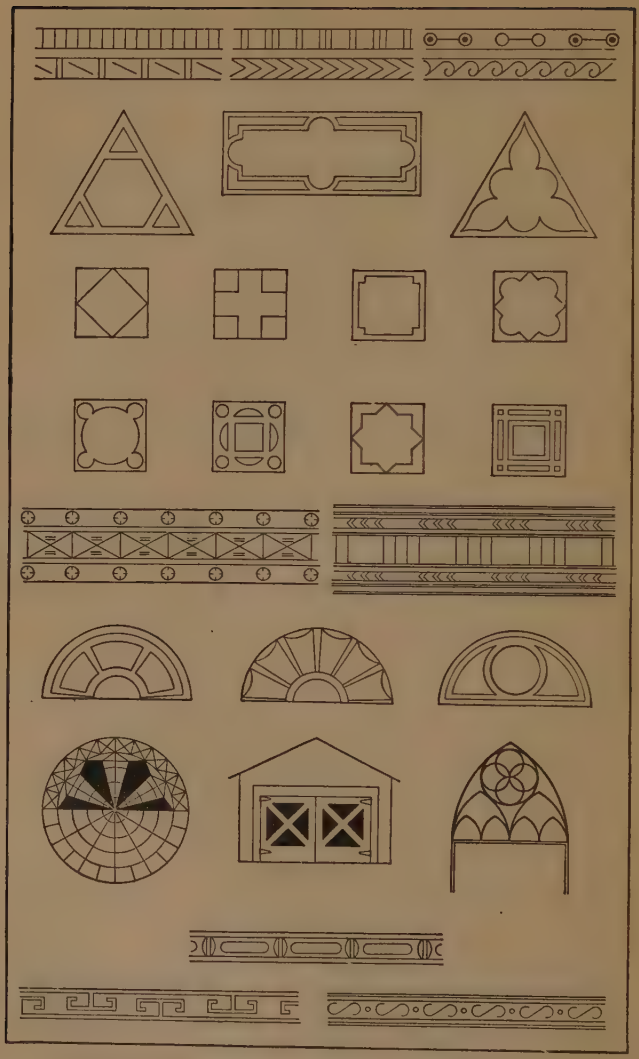
A brief sketch of the history of geometry is found on pages 383–388. This, however, will be better understood at the completion of the course.

ILLUSTRATIONS AND DEFINITIONS OF TERMS



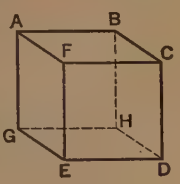
1. Physical Solid. Any *material* body occupies a certain definite portion of space. Such a material body may be called a *material* or *physical solid*. The blocks here represented are physical solids. They are definite parts of space filled with substance or matter.

2. Geometric Solid. Geometry does not deal with such physical solids, but with their *form* and *size*. If we have a wooden cube such as is shown here, we do not study the



GEOMETRIC FIGURES IN CONVENTIONAL DESIGNS

material (the wood), its texture, or its color. We study simply the *size* and *shape* of the figure as shown by the accompanying sketch, which represents a *geometric solid.* A physical or material solid may be seen and handled. A geometric solid is a *mental concept* or idea which can only be *represented.*



3. Geometric Surfaces. The limits or boundaries of a solid are *surfaces.* There are six surfaces shown in the figure in section 2 (§ 2). Name them.

4. Geometric Lines. The limits or boundaries of a surface are *lines.* Each of the six surfaces shown in § 2 has four lines as boundaries. Name the two surfaces of which *ED* is a boundary. What are the boundaries of *AFEG?*

5. Geometric Points. The limits or boundaries of a line are *points.* A point may also be considered as the *place* of intersection of two lines. How many points serve as the boundaries of the lines shown in § 2? Point *C* is one of the boundaries for what three lines?

6. Geometric Magnitudes. Solids, surfaces, lines, and points are called *geometric magnitudes.*

Oral Exercises

The *space* occupied by the vacuum in an electric light bulb represents what geometric magnitude? A *shadow* represents what magnitude? the *boundaries* of a shadow? the *corner* of a shadow?

7. Dimensions. A *solid* has extent in three principal directions; that is, it has three *dimensions—length, breadth,* and *thickness.* A *surface* has two dimensions—length and breadth. A *line* has one dimension—length. A *point* has no dimensions; it marks *position* only.

8. Generation of Geometric Magnitudes. We may obtain another point of view regarding what the fundamental geometric magnitudes are, by considering the following:

(1) If a *point* moves (except when it turns upon itself) it traces a *line;*

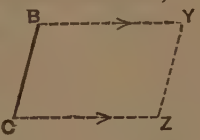
(2) If a *line* moves (except along itself) it traces a *surface;*

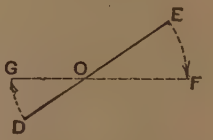
(3) If a *surface* moves (except along itself) it traces a *solid.*

For example:

(1) Point A generates the line AX;

(2) BC generates the surface $BYZC$; or, ED generates the surface $EFOGD$;

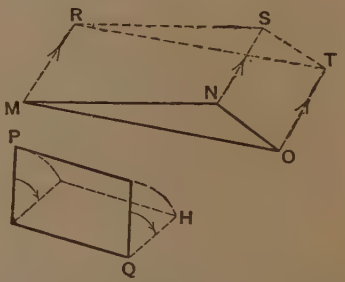




(3) Surface MNO generates the solid RON;

or,

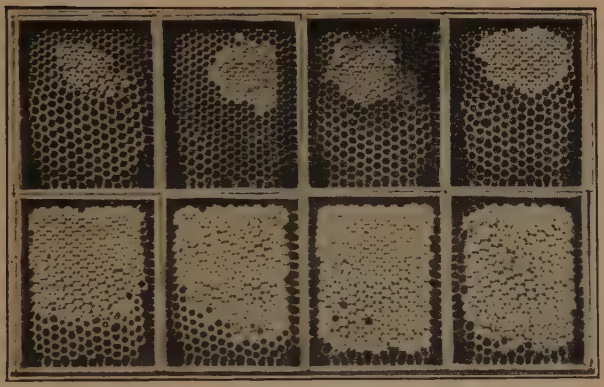
Surface PQ generates the solid PHQ.



9. Geometric Figure. A geometric figure, or simply a figure, is a *point, line, surface, solid,* or any *combination* of them.

10. Plane Figures. In plane geometry only such figures as can be represented in a *plane* (a smooth surface) are studied. These figures are made up of points and lines.

11. Representation of Points and Straight Lines. A point is represented by a small dot like a period, or by the intersection of two lines; as, points A and B. A straight line is represented by the mark made when drawing a pencil along a straightedge. (A straightedge may



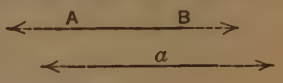
Courtesy, American Museum of Natural History

The storehouse of the honeybee is a series of hexagonal rooms.

Courtesy, Beatrice B. Bell

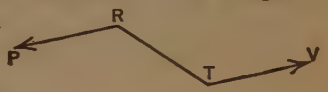
From the design of the web one might think that the spider understands geometry.

be made by folding a piece of paper into a sharp crease.) It is usually named by two of its points or by some letter placed upon it: as, line AB, and line a. The word *line* when used alone usually means a "straight line," since when any other line is meant it is so stated.

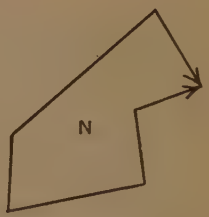


12. Curved Line. CD is a curved line. No part of it is straight.

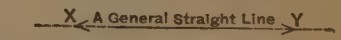
13. Broken Line. $PRTV$ is a broken line. It is composed of successive straight lines, which do not form a straight line.



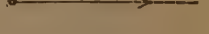
14. Closed Figure. When the ends of a curved or broken line meet, a closed figure is formed; as, figures M and N.



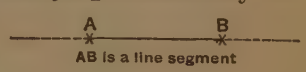
15. General Straight Line. A straight line is considered as extending indefinitely in *both* directions from any point in it. Such a line is the general or *unlimited* straight line.



16. Ray. A line beginning at a definite point and extending indefinitely in one direction from it is called a *ray*.



17. Line Segment. The part of a line lying between any two of its points is called a *segment* of the line or simply a *line segment*.



18. Equal Line Segments by Measurement. Two line segments are said to be equal, if they have the same *measure;*

that is, if their lengths can be expressed by the *same number of common units*.

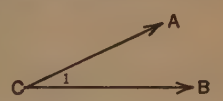
Oral Exercises

What special geometric figure is represented by the smooth polished surface of a plate glass mirror? by a ray of light? by the equator of the earth? by the edges of a series of saw teeth? by the path of a radio wave? by the following figures?

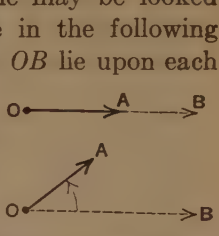
From what you can see in and about your classroom describe several things which may represent a line segment; a broken line; a closed figure; a point.

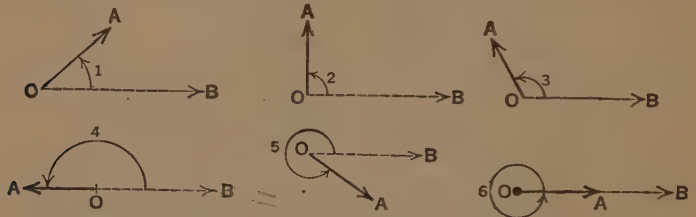
19. Representations of Ideal Magnitudes. If we sketch or draw a straight line, a curved line, a broken line, or a point, these sketches are only *representations* of the true or ideal geometric figures. Take for example a chalk line; it is really a *solid*. A pencil line, no matter how fine, is also a solid. Your photograph *represents* you; it is not really you. The sound which comes from the phonograph or radio is not really a human voice; it is the *representation* of it. So all the sketches or drawings we make are only representations of geometric or *ideal magnitudes*. They are like the cartoons of men that we see in the daily papers in that they emphasize and portray the *principal characteristics* of the subjects.

20. Angle. A plane angle ($\angle$) is the figure formed by two rays which have their end points in common. The rays are called the *sides* of the angle and the common point is called the *vertex* of the angle. Thus CA and CB are the sides of the angle BCA and C is its vertex. An angle is named usually in one of three ways; $\angle BCA$, $\angle C$, or $\angle 1$. Observe that the vertex (C) is placed in the middle when three letters are used to name an angle.



21. Generation of an Angle. An angle may be looked upon as having been *generated* or made in the following manner: Suppose that two rays OA and OB lie upon each other as shown in the figure, and OA is made to rotate about O and comes to rest as shown in the second figure. The size of the angle O will depend upon how much OA has been rotated from OB and not upon the lengths of the rays OA and OB. The following are angles of various sizes:

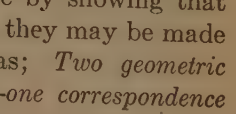




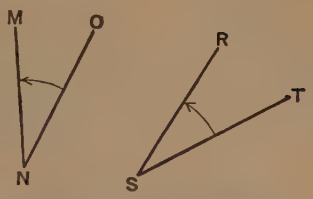
22. Equal Angles by Measurement. Two angles have the same size when they are the result of the *same amount* of *rotation*, which is usually measured in *degrees*. That is, they are equal if they have the same number of *common units of measure*.

23. Equal Angles and Equal Line Segments by Coincidence. Besides establishing the same number of common units of measure to determine the equality of two line segments or two angles, it is often desirable to do this without measuring them. This may be done by showing that their *limits* or *boundaries coincide*, or that they may be made to coincide. Sometimes it is stated as; *Two geometric magnitudes are equal if there is a one-to-one correspondence between their respective parts.*

Two line segments AB and CD are equal if when A is placed upon C, B falls upon D.



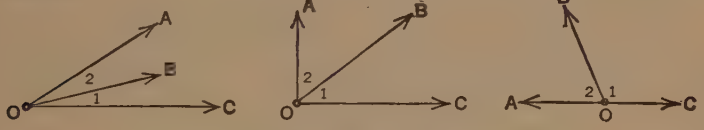
Two angles ONM and TSR are equal if, when NO is placed upon ST, NM falls along SR.



24. Two equal line segments can be made to coincide by matching their respective *end points*.

Two equal angles can be made to coincide by matching their *vertices* and the *positions* of their respective sides.

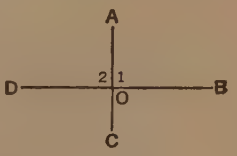
25. Adjacent Angles. Two angles which have a common vertex and a common side between them are called *adjacent* angles. In the following figures, $\angle 1$ and $\angle 2$ are adjacent angles.



Oral Exercises

In each figure of § 25 what is the common vertex? What is the common side? May two angles have a common vertex, but not a common side? How many angles may have a common side between them? How many angles may have a common vertex? Illustrate.

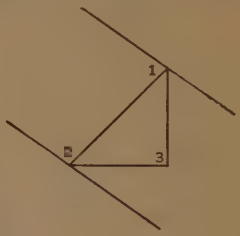
26. Perpendicular Lines. If two equal adjacent angles are formed when one straight line meets another, the lines are said to be *perpendicular* ($\perp$) to each other. Thus, if AC cuts DB at O so that $\angle 1 = \angle 2$, each of the lines AC and DB (or any part of either) is perpendicular to the other. That is, $AC \perp DB$ and $DB \perp AC$.



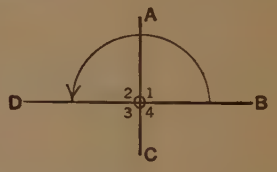
(The symbol $\perp$ is also read, "is perpendicular to.")

The pupil should remember: (1) **in order to prove two lines perpendicular, it must be shown that these lines form two equal adjacent angles;** (2) **that two perpendicular lines form two equal adjacent angles.**

27. Right Angle. An angle formed by two perpendicular lines is a *right angle.* In the following figures, angles 1, 2, 3, and 4 are right angles.



28. Straight Angle. An angle which is equal to the sum of two right angles is a *straight angle.* In the figure, if $AC \perp DB$, that is, if $\angle 1 = \angle 2 = \text{rt. } \angle$, then the angle BOD is a straight angle.



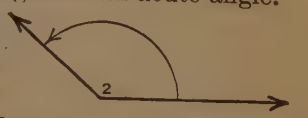
Observe that the sides of the straight angle BOD are OB and OD and that they lie in the same straight line DB. A straight angle may also be defined as one whose sides lie in the same straight line, but do not coincide.

29. Zero Angle. When two rays extend from the same point and coincide, the angle formed is called a *zero angle* if there has been no rotation of one ray from the other.

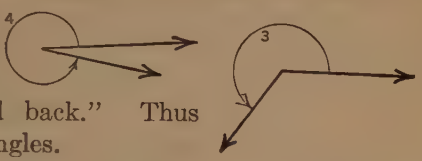
(May the figure of a zero angle represent any other angle if the amount of rotation is not known?)

30. Acute Angle. An angle whose size is between a zero angle and a right angle is called an *acute angle.* The word *acute* means "sharp." Thus, $\angle 1$ in the figure is an acute angle.

31. Obtuse Angle. An angle whose size is between a right angle and a straight angle is an *obtuse angle.* The word *obtuse* means "blunt." $\angle 2$ is an obtuse angle.

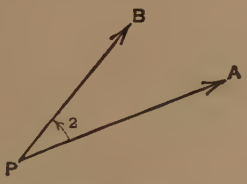


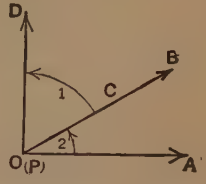
32. Reflex Angle. An angle whose size is between a straight angle and two straight angles is called a *reflex angle*. The word *reflex* means "to bend back." Thus $\angle 3$ and $\angle 4$ are reflex angles.



33. Complementary Angles. Two angles whose sum is equal to a right angle are called *complementary angles*. Each is the *complement* of the other. The word *complementary* is derived from the same word as *complete*. Two complementary angles *complete* or fill up a right angle. It is not necessary that the two angles be adjacent; all that is required by the definition is that their sum equals a right angle. One angle might be in New York and the other in San Francisco.

In the figure,





$\angle 1$ and $\angle 2$ are complementary if their sum as shown makes $\angle AOD$ a right angle.

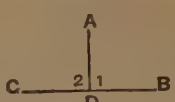
Oral Exercises

May an angle be of any size? Upon what does the size of an angle depend? How many angles are formed when two lines intersect through a point? Are all such angles equal? When are two lines said to be perpendicular? What is a straight angle? A right angle is one-half of what angle? What is the meaning of the word acute? May an acute angle and an obtuse angle together equal a straight angle? May two acute angles be complementary?

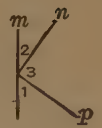
Sight Exercises

1. In the figure, what kind of angles are $\angle 1$ and $\angle 2$?

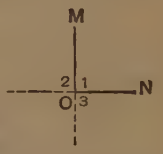
2. In the figure, if $\angle 1 = \angle 2$, what is the relationship of BC and AD? of CD and AD?



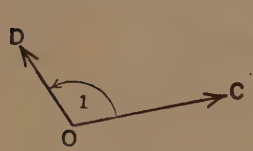
3. If $\angle 1 = \frac{2}{3}$ rt. $\angle$, $\angle 2 = \frac{1}{3}$ rt. $\angle$ and m is a straight line, what is the relationship of $\angle 1$ to $\angle 2$? How large is $\angle 3$?

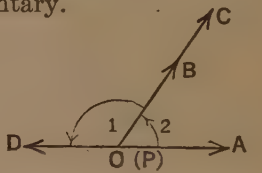


4. If $\angle 1 = $ rt. $\angle$, how large is $\angle 2$? $\angle 3$? Why? What is the relationship of NO to MO?



34. Supplementary Angles. Two angles whose sum is equal to a straight angle are called *supplementary angles*. Each is called the *supplement* of the other. The word *supplement* means "something added" which supplies a deficiency or a lack. There is no part of a straight angle lacking or missing when it is filled up by two angles which are supplementary.





$\angle 1$ and $\angle 2$ are supplementary if their sum as shown makes $\angle AOD$ equal to a straight angle.

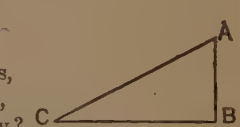
Oral Exercises

1. In the $\triangle ABC$, if

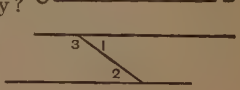
$\angle A + \angle B + \angle C = 2$ right angles,

and $\angle B = 1$ right angle,

are $\angle A$ and $\angle C$ complementary? Why?



2. In the adjoining figure, if $\angle 1 = \angle 2$ and $\angle 1 + \angle 3 =$ a straight angle, are $\angle 3$ and $\angle 2$ supplementary?



35. Vertical Angles. Two non-adjacent angles formed by two intersecting straight lines are called *vertical angles*. In the figure, $\angle 1$ and $\angle 2$ are vertical angles; $\angle 3$ and $\angle 4$ are vertical angles.

Oral Exercises

Are angles 1 and 3 adjacent angles? Why? Are they supplementary? Why? Are they vertical angles? Why? Name four pairs of supplementary angles. Name two pairs of equal angles. (See *Prin.* 27, p. 396.)

Exercises. Free-hand Drawing of Angles

1. Sketch an acute angle.
2. Sketch two adjacent angles.
3. Sketch two equal adjacent angles.
4. Sketch two equal adjacent angles which are complementary.
5. Sketch two equal adjacent angles which are supplementary.
6. Sketch two unequal complementary adjacent angles.
7. Sketch two unequal supplementary adjacent angles.
8. Sketch two equal angles and their complements.
9. Sketch two equal obtuse angles and their supplements.
10. Sketch an acute angle and an obtuse angle; their complements; their supplements.

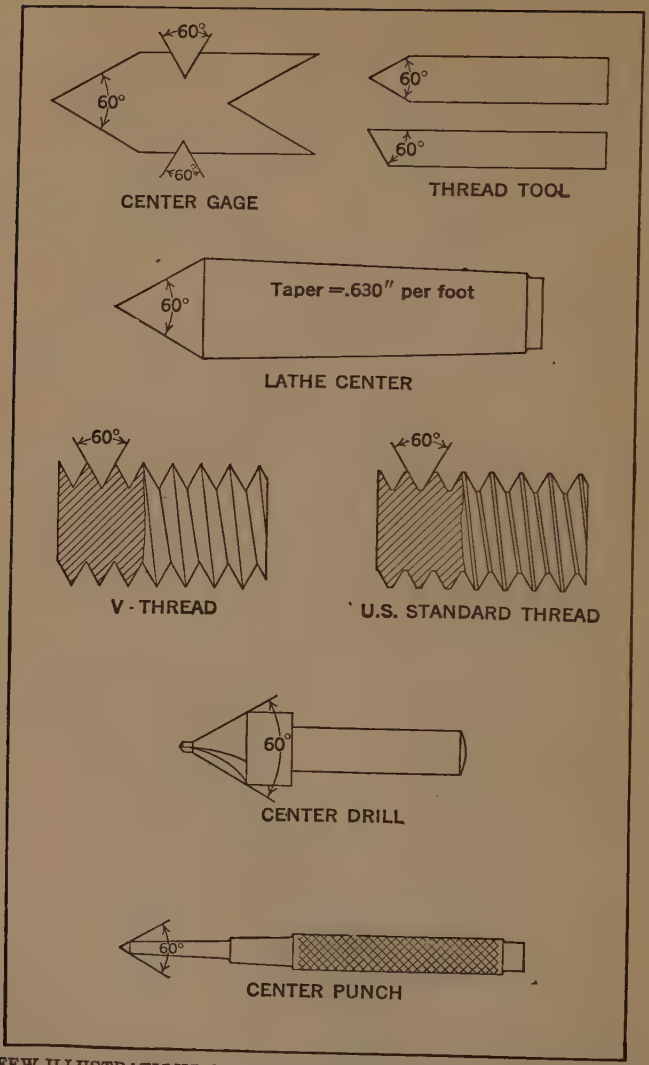
36. Units of Measure for Angles. Angles are measured by taking as a *unit* $\frac{1}{90}$ of a right angle. This unit is called an *angle degree*. Each degree is divided into 60 equal parts called *minutes*. Each minute is divided into 60 equal parts called *seconds*. That is

60 seconds (″) = 1 minute (′)
60 minutes = 1 degree (°)
90 degrees = 1 right angle
4 right angles = 1 perigon or 1 revolution.

The sum of all the angles about a point in a plane is a *perigon*, four right angles, or 360°.

Perigon is derived from two Greek words, *peri* meaning "around" and *gonia* meaning "angle."

Forty degrees, twenty minutes, ten seconds is written 40° 20′ 10″.



A FEW ILLUSTRATIONS OF THE 60° ANGLE IN THE MACHINE SHOP
Many problems of the machinist may be solved by applying facts learned in geometry.

Oral Exercises

How many degrees in a straight angle? In two straight angles? In an angle of one revolution? 270° is how many right angles? 45°, 30°, and 22° 30′ are each what part of a right angle?

Exercises on Angles

1. Find the complement of 30°; of 45°; of 19° 15′; of 25° 20′.

2. What is the supplement of an angle of 90°? of 45°? of 30° 30′? of 100° 50′?

3. Find two angles whose sum is 90° such that one angle is twice the other.

HINT: $x + 2x = 90°$.

4. One angle is four times another and they are supplementary. Find each angle.

5. Find three angles whose sum is 180° and whose ratio is 1:2:3.

HINT: $x + 2x + 3x = 180°$.

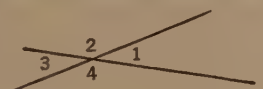
6. Find the angle whose complement is 70° more than the angle itself.

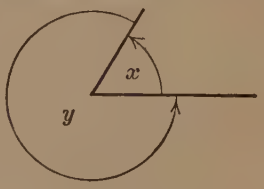
HINT: $x + (x + 70) = 90°$.

7. Find the supplement of an angle which is 120° less than the angle.

8. How many degrees are between the hands of a clock at four o'clock? (There are two such angles.) How many degrees between them at 7:45?

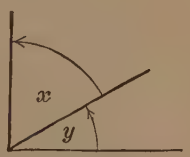
9. In the figure, if $\angle 1 = 30°$, find $\angle 2$, $\angle 3$, and $\angle 4$.





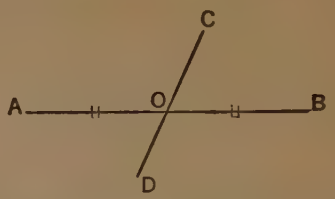
10. In the figure, if $\angle x + \angle y = 360°$, and $4\angle x = \angle y$, find $\angle x$.

11. In the figure, if $\angle x + \angle y = 90°$, and $\angle x - \angle y = 18°$, find $\angle x$ and $\angle y$.



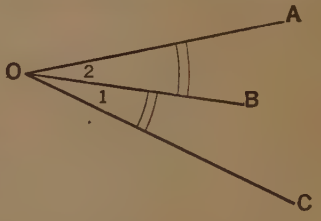
37. Bisectors of Lines and Angles. A line or an angle is *bisected* when it is divided into two equal parts. The *mid-point* of a line, or any line which passes through the mid-point is called the *bisector* of the line. The bisector of an angle is the line which passes through the vertex and which with the vertex forms the *common boundary* of two equal adjacent angles.

In the figures,



if $OA = OB = \frac{1}{2} AB$, point O is a bisector and CD is also a bisector of AB.

If $\angle 1 = \angle 2 = \frac{1}{2} \angle COA$, OB is the bisector of angle COA.



The pupil should remember that **the bisector of a line makes two equal lines** and that **the bisector of an angle makes two equal angles.**

NOTE: In the figures above, equal parts are marked alike. Observe that such a scheme aids the eye in knowing in the first case that $OA = OB = \frac{1}{2} AB$ and in the second, that $\angle 1 = \angle 2 = \frac{1}{2} \angle COA$.

38. Perpendicular Bisector of a Line. When a line is perpendicular to another at its mid-point, it is called the *perpendicular bisector* of the line.

In the figure, AO answers these *two* conditions and is called the perpendicular bisector of CB.

The pupil should remember that **a perpendicular bisector makes two equal adjacent angles and two equal lines.**

39. Requirements of a Good Definition. It is not necessary to state the definition of a term in the words of this book, but it may be done in the pupil's own language. A good definition, however, should embody four points. They are:

(1) the term defined must be described in better known terms than the thing itself;

(2) such description must limit the term to the thing itself;

(3) the statement must not be redundant;

(4) the definition must be reversible.

As a pupil once expressed it, "A definition must be in easy words which say just enough and not too much, and you must be able to turn the whole thing around."

Since pupils usually have more or less difficulty in forming good definitions, it is suggested that the more important of the definitions be committed to memory.

(A list of terms with the dictionary form of definition is found on pages 388–394.)

Review Exercises

1. Sketch the bisectors of a pair of complementary adjacent angles.

2. Sketch the bisectors of a pair of supplementary adjacent angles. Do these bisectors look to be perpendicular to each other?

3. Draw a line 2 in. long. Locate its mid-point. How many lines may be drawn through it bisecting the given line? How many perpendicular bisectors?

4. Draw two equal angles. Sketch their bisectors. What can be said of the four angles thus formed?

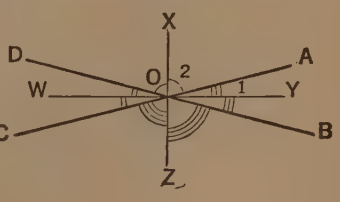
5. Draw a pair of vertical angles. Sketch their bisectors. Do they appear to lie in the same straight line?

6. How many lines may be called the bisector of a given line?

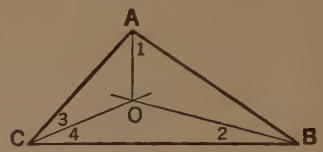
7. How many bisectors has a given angle?

8. An angle of 75° is bisected. What is the size of each half?

9. In the figure, OX, OY, OZ, and OW are bisectors of the respective angles. How large is $\angle YOX$?



10. In the figure, OA, OB, and OC are bisectors and $\angle A + \angle B + \angle C = 180°$. How large is $\angle 1 + \angle 2 + \angle 3$?



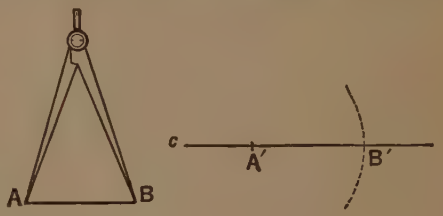
11. Draw any angle. Sketch its bisector. Name the parts $\angle 1$ and $\angle 2$. What is the relation of $\angle 1$ to $\angle 2$? Why?

12. Draw any line segment and call it XY. Represent its midpoint by S. What equality is known? Why?

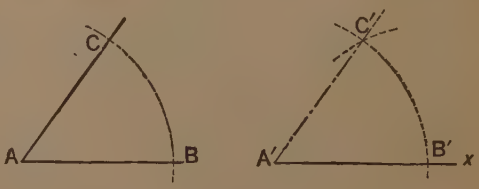
13. Draw any line such as XY and sketch its perpendicular bisector. What two equalities are known? Why?

40. Drawing with Straightedge and Compasses. Up to this point the pupil has sketched (or drawn free-hand) certain figures. Their construction by means of instruments will now be explained.

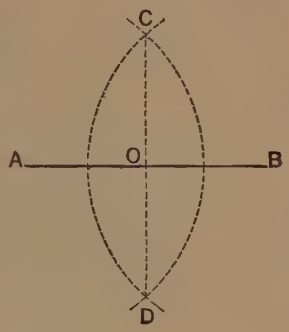
1. To Draw a Line Equal to a Given Line. The pupil already understands how to measure a line in inches and to draw a second line equal in measurement to the first. In formal geometry a line is not measured with a ruler, but with the *compasses*. If AB is a given line, adjust the compasses so that its points coincide with A and B. With point A' on line c as a center, draw an arc cutting c at B'. Then $A'B' = AB$.



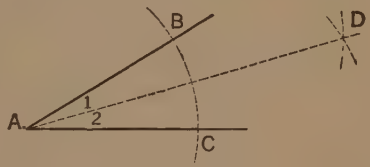
2. To Draw an Angle Equal to a Given Angle. With the vertex of the given angle ($\angle A$) as a center and with any convenient opening of the compasses draw an arc intersecting the sides of the angle at B and C. With another given point (A' on line x) as a center and with the same opening of the compasses draw an arc ($B'C'$). Adjust the compasses to make its points coincide with B and C. With B' as a center draw another arc intersecting arc $B'C'$ at C'. Draw $A'C'$. Then $\angle A' = \angle A$.



3. To Bisect a Given Line. A line can be bisected approximately by first measuring its length in inches and then marking a segment equal to one-half its measured length. Greater accuracy, however, may be obtained by the use of compasses. To bisect a line (AB) place the sharp point of the compasses on A as a center and draw an arc sufficiently long to intersect at C and D another arc similarly drawn with B as a center. Then carefully draw a straight line through C and D. CD is the perpendicular bisector of AB.

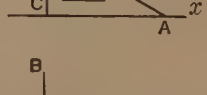


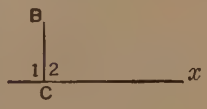
4. To Bisect a Given Angle. To bisect an angle ($\angle A$) place the sharp point of the compasses on A as a center and draw an arc intersecting the sides of the angle at B and C. With B and C as centers, draw arcs intersecting at D. Draw AD. Then $\angle 1 = \angle 2 = \frac{1}{2}\angle A$. With practice the pupil will find it possible to obtain the necessary points of intersection by drawing shorter arcs.

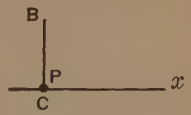


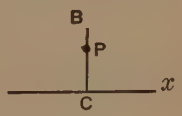
5. To Draw Perpendicular Lines. The compasses can be used to find a line perpendicular to a given line as in Construction 3 where $CD \perp AB$.

Perpendiculars can also be drawn by the use of an instrument called the *draughtsman's triangle*. This triangle is usually made of celluloid with $\angle A = 30°$, $\angle B = 60°$, and $\angle C = 90°$. To draw a line perpendicular to a given line (x) place the triangle with one edge along x as shown in the figure and draw a line along BC. Then $BC \perp x$ and $\angle 1 = \angle 2 =$ rt. $\angle$. If a perpendicular is to be drawn to line x through a point (P) on the given line, slide the triangle along x until C coincides with P. If a perpendicular is to be drawn to line x through a point (P) not in the given line, slide the triangle along x until P lies along the edge BC.









In place of the draughtsman's triangle, the carpenter uses a *try-square* in the same way. The pupil may take for a try-square two adjoining edges of a stiff card or of a ruler.

The five preceding constructions are fundamental. With their use the pupil may draw figures that are neat and accurate.

Drawing Exercises

1. Use an unmarked straightedge to draw two *unequal* lines of *unknown* length. Bisect one of them by using the compasses and bisect the other by measuring its length with a ruler and calculating its half. Which of the two methods is the more accurate and practical?

2. Draw a right angle and bisect it.

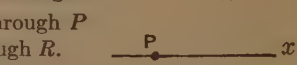
3. Find an angle which is one-fourth of a right angle. What is its size in degrees?

4. Bisect each of two supplementary adjacent angles. What kind of angle is formed by the bisectors?

5. Draw the supplement of a given acute angle.

6. Draw the complement of a given acute angle.

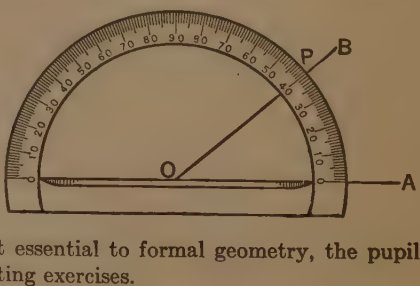
7. Draw a line perpendicular to x through P and another line perpendicular to x through R.



8. Draw a right angle and draw another angle equal to it by using Construction 2. Check the correctness of your construction.

41. Measurement of Angles by a Protractor.

An instrument called a *protractor* is used to measure the number of degrees in an angle. It is marked from 0° to 90° in both directions. The figure shows a protractor whose center O is at the vertex of $\angle AOB$ and whose 0° point is on OA. The reading at point P where OB crosses the protractor gives the size of the angle; 40° in this case.



While the protractor is not essential to formal geometry, the pupil may apply it to many interesting exercises.

Exercises with a Protractor

1. Measure with a protractor $\angle 1$ in § 33, $\angle 2$ in § 33, $\angle 1$ in § 34, $\angle 1$ and $\angle 2$ in § 35, $\angle 1$ and $\angle 2$ in § 37.

2. Draw two intersecting lines and measure each angle of a pair of vertical angles.

3. Draw two lines which meet to form adjacent angles. Measure each of the adjacent angles. What should their sum equal? What is the arithmetic sum of the angles in your figure?

4. Draw any angle and then see how closely you can estimate its size before measuring it with a protractor.

5. Draw with a protractor two 60° angles. Can the same method be used for drawing any two equal angles?

6. Draw any angle using a ruler only. Draw an angle equal to it by means of a protractor.

SOME FUNDAMENTAL FACTS

42. In your study of algebra, you have used the fact that (1) *If equals are added to equals, the sums are equal.* Such an accepted fact is called an *axiom.*

It may be illustrated by the following:

$$\begin{array}{r} 18 + 12 = 16 + 14 \\ 5 + 3 = 4 + 4 \\ \hline 23 + 15 = 20 + 18 \end{array} \qquad \begin{array}{r} a = b \\ c = d \\ \hline a + c = b + d \end{array}$$

Other axioms which you may recall are:

(2) *If equals are subtracted from equals, the differences are equal.*

(3) *If equals are multiplied by equals, the products are equal.*

(4) *If equals are divided by equals, the quotients are equal.*

(Division by zero not allowed.)

(5) *The whole of anything is equal to the sum of its parts.*

(6) *The whole is greater than any of its parts.*

There are other facts whose truths you have accepted without proof. Such as the following statements *A*, *B*, and *C:*

A) *Like numbers may be added in any order.*

B) Dividing one factor of a product by a number, divides the product by that number.

C) Dividing both dividend and divisor by the same number does not change the value of the result.

A full list of those *facts* which may be accepted without proof is given under the *Principles of Geometry*, pp. 396–399.

Exercises. Principles of Geometry

The following exercises (6 to 20) illustrate the meaning of the more important Principles, Nos. 1 to 9. In each case the pupil should state in his own words the principle involved before he verifies it from the list found on page 396.

1. Why do 3 ft. + 4 ft. = 4 ft. + 3 ft.? (Statement *A*, p. 25.)

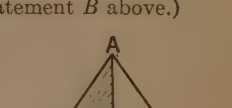
2. $a + b = b + a$. Why?

3. $\angle A + \angle B = \angle B + \angle A$. Why?

4. Why does $\frac{a^2 + 2ab + b^2}{a^2 - b^2} = \frac{a + b}{a - b}$? (Statement *C* above.)

5. Does $\frac{h(B - b)}{2} = \frac{h}{2}(B - b)$? Why? (Statement *B* above.)

6. $4 = 4$; $\angle C = \angle C$; $AB = AB$. Why? (*Prin.* 1, p. 396.)



7. In the figure, why does $AD = AD$? $\angle C = \angle C$? $AC = AC$?

8. If $x = y$ and $x = z$, why does $y = z$? (*Prin.* 3.)

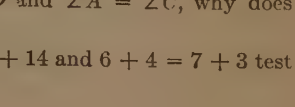
9. In the figure of No. 7, if $AD = AC$ and $AC = CD$, why does $AD = CD$?

10. In the same figure, if $\angle A = \angle D$ and $\angle A = \angle C$, why does $\angle D = \angle C$?

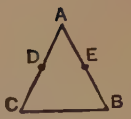
11. With the expressions $18 + 12 = 16 + 14$ and $6 + 4 = 7 + 3$ test *Prin.* 5.

12. With expressions $15 + 3 = 4 + 14$ and $5 + 2 = 7$ test *Prin.* 6.

13. In the figure, if $\angle ABC = \angle BCA$ and $\angle DBC = \angle BCD$, why does $\angle 1 = \angle 2$? (*Prin.* 6.)

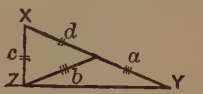


14. In the figure, if $AB = AC$ and $AE = AD$, what is true of CD and BE? Why?



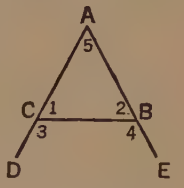
15. In the same figure, assume $AD = AE$ and $CD = BE$. What is true of AC and AB? Why? (*Prin.* 5.)

16. In the same figure, suppose that AC and AB are equal and are bisected by points D and E. Why does $AD = EB$? (As a special case under *Prin.* 8, the statement may be made, **"Halves of equals are equal."**



17. In the figure, if $a = b$ and $c = d$, then $b + c = a + d$. (*Prin.* 5.)

18. In the same figure, why does $XY = a + d$? (*Prin.* 4.)



19. In the figure, what kind of angles are DCA and ABE? If $\angle DCA = \angle ABE$ and $\angle 1 = \angle 2$, why does $\angle 3 = \angle 4$? (*Prin.* 6.)

20. In the same figure, if $\angle 1 = \angle 2$ and $\angle 2 = \angle 5$, why does $\angle 1 = \angle 5$?

43. There are two other classes of assumptions besides axioms which are listed under the Principles of Geometry—*postulates*, and *simply derived theorems*. Postulates refer to *geometric* magnitudes, while the axioms refer to *all* magnitudes, e.g., *numbers*.

Principle No. 37 which states that, *any geometric figure may be moved from one position to another without change either of shape or of size* is sometimes called the *Postulate of Mobility*. Another postulate (*Prin.* 14), called the *Postulate of Parallels* states that, *through a point one line and only one can be drawn parallel to a given line*.

Principle No. 25 states that *an angle can be bisected*, that is, any angle can be divided into two equal parts. This may seem to be a useless statement until one knows that any angle cannot be divided into any number of equal parts. An angle cannot be divided, for example, into three equal parts. Constructions in geometry are limited to those which can be made with *compasses* and an *unmarked*

straightedge. No way has been discovered as yet to *trisect* any given angle with these drawing instruments alone. Special angles, such as the *right* angle, can be trisected. The Principle refers to angles whose size is not known—to any angle—not a particular kind.

Some of the Principles, such as Nos. 27 and 39, can be proved to be true. Some of those marked (*) are often discussed or proved informally. However, we may take any of the Principles to explain why some statement made is true.

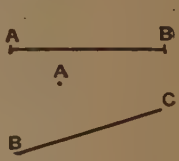
Exercises. Principles of Geometry

The following exercises illustrate the meaning of some of the more important Principles, Nos. 10–39. In each case the pupil should formulate in his own words the principle involved before he verifies it from the list found on pages 396–398.

1. X is a point. How many lines may be drawn through it? (*Prin.* 10.)

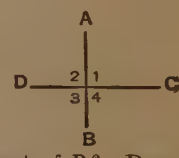
2. How many points determine a straight line? (*Prin.* 13.)

3. Why are there two sights on a gun?



4. Given the line segment AB. It may be produced through B for 1 in. and through A for 12.57 in. Why? (*Prin.* 11.)

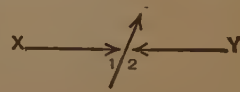
5. Through point A how many lines may be drawn parallel to BC? (*Prin.* 14.)



6. In the figure, if $\angle 1 = \text{rt.}\angle$ and AB and CD are straight lines, why does $\angle 2 = \text{rt.}\angle$?

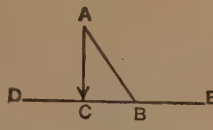
7. In the same figure, give two reasons why $\angle 1 = \angle 3$. (*Prins.* 27 and 28.)

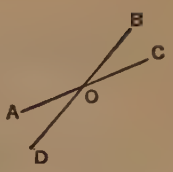
8. Given two equal acute angles, A and B. Why is the complement of A equal to the complement of B? Draw the angles.



9. In the figure, the lines meet in a point and $\angle 1 + \angle 2 = 180°$. XY is a straight line. (*Prin.* 30.)

10. In the adjoining figure, if $AC \perp DE$, then AB is not perpendicular to DE. (*Prin.* 15.)

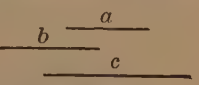




11. In the same figure, the distance from A to DE is AC and is not AB. (*Prin.* 23.)

12. AC and DB are two straight lines which intersect at O. Can they intersect at any other point? (*Prin.* 20.)

13. If $a \parallel b$ and $a \parallel c$, then $b \parallel c$. (*Prin.* 16.)



PROOF. ILLUSTRATIONS OF WHAT IT MEANS

44. The word *proof* is defined as, "*a process of reasoning that leads to an absolutely certain conclusion.*" It is the establishment of a *truth* by the use of *other truths.* In geometry, it is the system of statements (each supported by a *fact*) which leads to a *new* truth.

Such a process is not new to the pupil as may be seen from the following examples:

Example 1. If 4 books cost \$12, what do 5 books cost?

Solution:

Statements	Facts
1. 4 books = \$12.	1. Given.
2. 1 book = \$ 3.	2. One book costs $\frac{1}{4}$ as much as four books.
3. 5 books = \$15.	3. Five books cost 5 times as much as one book.

Example 2. If $\frac{x}{2} + 3 = 7$, find x.

Solution:

Statements	Facts
1. $\frac{x}{2} + 3 = 7.$	1. Given.
2. $\frac{x}{2} = 4.$	2. By the Axiom of Subtraction, 3 is subtracted from each side.
3. $x = 8$	3. By the Axiom of Multiplication, each side is multiplied by 2.

Unless we have made some error in the computations involved, which of course is not a part of the argument at all, we are certain that 5 books cost \$15 in the first example and that $x = 8$ in the second.

The preceding two groups of exercises (pages 26–29), which were used to illustrate the various principles, were *one-step* proofs. In each case, the conclusion was supported by a fact (the *principle*).

45. Theorem, Hypothesis, and Conclusion. A statement whose truth admits of proof is called a *theorem*. Every theorem may be expressed by a sentence which has one clause beginning with *if* and a second clause beginning with *then*. The *if clause* contains the special condition which is *given or assumed* to be true; while the *then clause* contains that which is *to be proved*. The conditional clause is called the *hypothesis;* the truth to be proved is called the *conclusion*. For example: "If a triangle is equilateral, it is equiangular" is a *theorem*. "If a triangle is equilateral" is the *hypothesis;* "it is equiangular" is the *conclusion*. The hypothesis states that the triangle to be considered has its three *sides* equal. The conclusion states that its three *angles* are equal.

Oral Exercises

Point out the hypothesis and the conclusion in each of the following exercises, remembering that the hypothesis is the given or assumed part of a theorem and that the conclusion is the part to be proved:

1. If $A = B$, then $C = D$.
2. If two angles are each 70°, they are equal.
3. You must pay attention, in order to understand.
4. If you attend your geometry class more regularly, you will understand your work better.
5. If $x = y$ and $y = z$, then $x = z$.

6. If two adjacent angles have their exterior sides in the same straight line, they are supplementary.

7. Straight angles are equal.

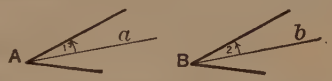
8. In an isosceles triangle the angles opposite the equal sides are equal.

9. The sum of the angles of a triangle is equal to a straight angle.

Exercises. Informal Proofs

In each of the following exercises state the hypothesis, the conclusion, and the proof:

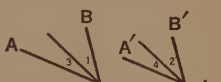
1. If $\angle A = \angle B$ and the angles are bisected, then $\angle 1 = \angle 2$.



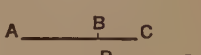
Solution: The *hypothesis* is that $\angle A = \angle B$ and that a and b are their bisectors. The *conclusion* is that $\angle 1 = \angle 2$. The *proof* is that

1. $\angle A = \angle B$.	1. By hypothesis.
2. $\angle 1 = \angle 2$.	2. Because halves of equals are equal.

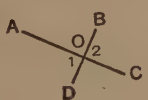
2. If $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$, then $\angle BOA = \angle B'O'A'$. (Hyp. and *Prin.* 5.)



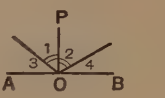
3. If $AC = PS$ and $AB = PR$, then $BC = RS$. (Hyp. and *Prin.* 6.)



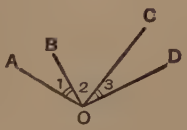
4. In the figure, $\angle 2$ is a right angle, if $\angle 1$ is a right angle.



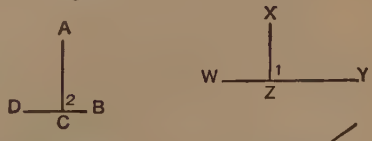
5. If $PO \perp AB$ and $\angle 1 = \angle 2$, prove by using Hyp. and *Prin.* 28 that $\angle 3 = \angle 4$.



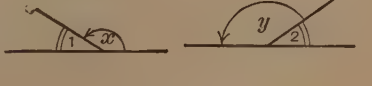
6. If $\angle 1 + \angle 2 = 90°$ and $\angle 1 = \angle 3$, then $\angle 2 + \angle 3 = 90°$. (Hyp. and *Prin.* 3.)



7. If $AC \perp DB$ and $XZ \perp WY$, then $\angle 1 = \angle 2$. (Hyp., definition of right angle, and *Prin.* 26.)

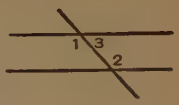


8. The supplements of equal angles ($\angle 1$ and $\angle 2$), are equal.



HINT:

$\angle 1 + \angle x = \angle 2 + \angle y.$	Why?
$\angle 1 = \angle 2.$	Why?
$\angle x = \angle y.$	Why?



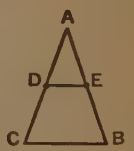
9. If $\angle 1 = \angle 2$ and $\angle 1 + \angle 3 = 180°$, then $\angle 3 + \angle 2 = 180°$.

10. In the figure, if $\angle 1 = \angle 3$, then $\angle 2 = \angle 4$.

HINT: The statements of the proof are

1. $\angle 1 + \angle 2 =$ st. angle.	1. Why?
2. $\angle 3 + \angle 4 =$ st. angle.	2. Why?
3. $\angle 1 + \angle 2 = \angle 3 + \angle 4.$	3. Why?
4. $\angle 1 = \angle 3.$	4. Why?
5. $\angle 2 = \angle 4.$	5. Why?

11. A line which bisects two equal line segments from the same point cuts off two other equal line segments from that point.



Given: $AC = AB, AD = DC = \frac{1}{2}AC, AE = EB = \frac{1}{2}AB.$

To prove: $AD = AE.$

Proof: (The proof is left to the student.)

HINT: AD and AE are halves of equals.

FORMAL ARRANGEMENT OF A WRITTEN DEMONSTRATION

46. Parts of a Written Demonstration. Every complete demonstration of a theorem, when arranged in a formal manner, comprises the following five parts and in this order:

I. Statement of the *theorem* in full.

II. A *figure.*

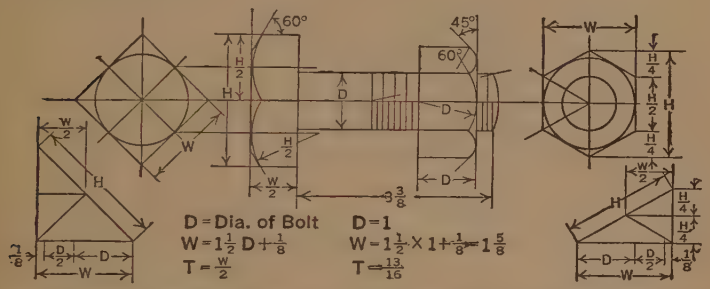
III. Statement of the *hypothesis.*

IV. Statement of the *conclusion.*

V. An orderly *proof.*

47. The theorem is written as a complete sentence. It may be used later, *after it has been established by proof*, as a *supporting fact.* Such statements, therefore, should be *thoroughly understood and memorized.*

48. The figure is a sketch or drawing of the points, lines, angles, or other geometric figures under discussion. It represents *any* and hence *every* such figure. Any suitable letters or characters may be used to aid in describing and reading the figure.



That mechanical drawing depends upon an understanding of geometric figures may be seen even in such a simple drawing as shown.

49. The hypothesis states, in terms of the letters or characters on the figure, what the theorem *assumes* to be true. This is usually stated under a title called **Given.**

50. The conclusion states, in terms of the letters or characters on the figure, what the theorem requires to be *proved.* This appears under a title called **To prove.**

The statements under **Given** and **To prove** are more conveniently written by using abbreviations and symbols. Full lists of these are found on page xv.

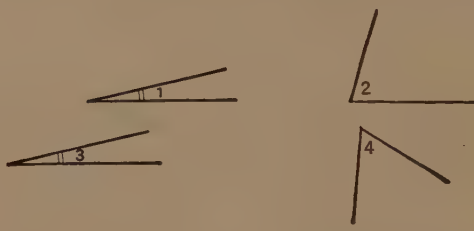
51. The proof consists of two parts: the series of statements of the argument, and the corresponding series of supporting facts. These facts *will always contain* among them *the hypothesis* definitely stated or implied, and may contain any statements previously agreed upon, such as *definitions, principles, theorems,* and any combination of them. *Every* statement in the development of the argument must be supported by such *agreed upon facts* and by no others. The

last statement of the proof should be the same as the statement of the *conclusion.*

Many times the correctness of the conclusion may be quite evident to the pupil. *The proofs of geometry,* for the most part, *are not given to show that certain facts are correct, but that these facts are the natural outgrowth of the special conditions of the theorem when combined with what has been previously known.*

ILLUSTRATIVE DEMONSTRATIONS

52. Theorem. *Complements of equal angles are equal.*



Given: $\angle 1 = \angle 3$, $\angle 1 + \angle 2 = 90°$, $\angle 3 + \angle 4 = 90°$.

To prove: $\angle 2 = \angle 4$.

Proof:

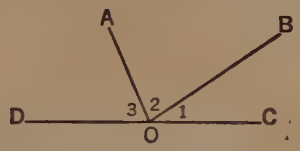
STATEMENTS	FACTS
1. $\angle 1 + \angle 2 = 90°$.	1. Given.
2. $\angle 3 + \angle 4 = 90°$.	2. Given.
3. $\angle 1 + \angle 2 = \angle 3 + \angle 4$.	3. Substitution.
4. $\angle 1 = \angle 3$.	4. Given.
5. $\angle 2 = \angle 4$.	5. Axiom of Subtraction.

53. Theorem. *Supplements of equal angles are equal.*

(The proof of this theorem is similar to the proof of § 52 and should be made by the pupil.)

NOTE: §§ 52 and 53 constitute the proof of Principle 28, p. 397.

54. Theorem. *The sum of the angles about a point on the same side of a straight line equals a straight angle.*



Given: DC is a straight line and $\angle 1$, $\angle 2$, and $\angle 3$ are all angles with O as a vertex.

To prove: $\angle 1 + \angle 2 + \angle 3 = \text{st. } \angle$.

HINT: Use Principle 4.

Proof:

STATEMENTS	FACTS
1. DOC is a straight line.	1. Given.
2. $\angle DOC$ is a st. $\angle$.	2. § 28. (Not *Prin.* 28.)
3. $\angle DOC = \angle 1 + \angle 2 + \angle 3$.	3. A magnitude equals the sum of its parts. (*Prin.* 4.)
4. $\angle 1 + \angle 2 + \angle 3 = \text{st. } \angle$.	4. Sub.

Practice Proof

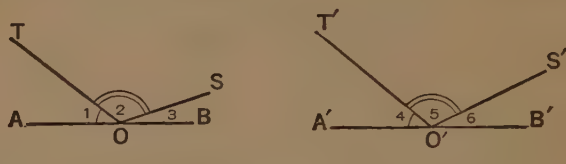
Theorem. *If a line is drawn through the vertex of a right angle and lies outside the angle, then the sum of two new angles thus formed is a right angle.*

HINT: Adapt the figure of § 54 to this theorem. What change is necessary?

$$\begin{array}{rrl} \angle 1 + \angle 2 + \angle 3 & = ? & \text{Why?} \\ \angle 2 & = ? & \text{Why?} \\ \hline \angle 1 \quad + \angle 3 & = ? & \text{Why?} \end{array}$$

Write out the full formal demonstration following the form shown in the Illustrative Demonstrations.

55. Sometimes when the hypothesis and the conclusion can not be given easily in a sentence, the statement of the theorem is omitted.



Given: AOB and $A'O'B'$ are st. lines and $\angle 1 = \angle 4$ and $\angle 2 = \angle 5$.

To prove: $\angle 3 = \angle 6$.

Proof:

STATEMENTS	FACTS
1. AOB and $A'O'B'$ are st. lines.	1. Given.
2. $\angle 1 + \angle 2 + \angle 3 = 180°$. $\angle 4 + \angle 5 + \angle 6 = 180°$.	2. Theorem immediately preceding. (§ 54.)
3. $\angle 1 + \angle 2 + \angle 3 = \angle 4 + \angle 5 + \angle 6$.	3. Sub.
4. $\angle 1 = \angle 4$, and $\angle 2 = \angle 5$.	4. Given.
5. $\angle 1 + \angle 2 = \angle 4 + \angle 5$.	5. Axiom of Addition.
6. By subtracting Statement 5 from Statement 3, $\angle 3 = \angle 6$.	6. Axiom of Subtraction.

Oral Exercises

1. What are the five essential parts of a written demonstration?

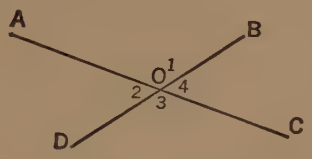
2. What is a proof? What are the names of its two parts?

3. What special kinds of statements may occur in the facts of a proof?

4. What is the meaning of the word *substitution*? What is its abbreviation?

5. What are vertical angles?

56. Theorem. *Vertical angles are equal.*



Given: AC and DB are straight lines.

To prove: $\angle 4 = \angle 2$ and $\angle 1 = \angle 3$.

HINT: Supplements of the same angle are equal.

Proof: STATEMENTS	FACTS
1. AOC is a straight line.	1. Given.
2. $\angle 1 + \angle 4 = 180°$.	2. If two adjacent $\measuredangle$s have their ext. sides in a st. line, they are supplementary. (*Prin.* 29.)
3. BOD is a straight line.	3. Given.
4. $\angle 1 + \angle 2 = 180°$.	4. *Prin.* 29.
5. $\angle 4 = \angle 2$.	5. Supplements of the same angle are equal. (*Prin.* 28.)
6. $\angle 4 + \angle 3 = 180°$.	6. *Prin.* 29.
7. $\angle 1 = \angle 3$.	7. Using Statements 2 and 6 and Principle 28.

NOTE: The above is the proof of Principle 27, p. 397.

Oral Exercises

1. In the figure of § 56, if $\angle 1 = \angle 4 =$ rt. $\angle$, what angles are equal? What angles are supplementary?

2. In the same figure, if $\angle 1 = 2\angle 4$, how many degrees in each of the four angles?

3. In the same figure, if $\angle 1 = 20°\ 30'$, find the three remaining angles.

Practice Proofs

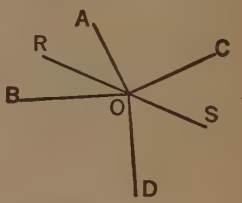
By following the form shown in the Illustrative Demonstrations prove the following exercises:

1. The bisector of one of two vertical angles bisects the other.

HINT: Sketch a pair of vertical angles. Draw the bisector of one of them. Prolong it through the common vertex. (*Prin.* 11.) Within the given pair of vertical angles are two new pairs of vertical angles. How are the angles of each pair related?

2. The angle formed by the bisectors of a pair of supplementary adjacent angles is a right angle.

HINT: Sketch a pair of supplementary adjacent angles. (See §§ 34 and 25.) Draw the bisectors of these angles. Number the angles thus formed. What pairs of angles are equal? Use § 54 and *Prin.* 3 to aid in the proof.



3. In the figure, $CO \perp OA$, $DO \perp OB$, and RO bisects $\angle AOB$. Prove $\angle COR = \angle ROD$.

HINT: Use the Axiom of Addition.

4. In the figure of Ex. 3, if $CO \perp OA$, $DO \perp OB$, and ROS bisects $\angle AOB$, prove $\angle ROB$ and $\angle SOC$ complementary.

HINT: Prove $\angle AOR + \angle SOC = 90°$. $\angle AOR$ equals what angle? Why? Use substitution to complete the proof.

5. The exterior sides of supplementary adjacent angles form a straight line. (This is Principle 30.)

HINT: Sketch the figure. Which two lines must be shown to lie in the same straight line?

6. Make up an exercise of your own. Sketch the required figure. State the "Given" and "To prove" in terms of the letters you place upon your figure. Write out a neat and carefully arranged "Proof."

CHAPTER ONE

RECTILINEAR FIGURES

57. Key to the Lettering of the Propositions.

C Standard Book Propositions required by the College Entrance Examination Board.

c Auxiliary Book Propositions required by the College Entrance Examination Board.

N Fundamental Propositions recommended by the National Committee.

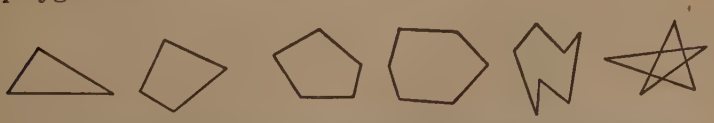
n Subsidiary Propositions recommended by the National Committee.

Propositions which are not designated by any letters have been added for coherence and additional information. Propositions are theorems or problems.

TRIANGLES

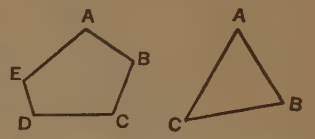
Our work thus far has dealt principally with angles, which are open figures. Whatever facts we have already learned about angles may also be used when angles are found in closed figures, the study of which we will now take up.

58. Polygon. A polygon is a closed broken line. The word polygon comes from two Greek words, *poly* meaning "many" and *gonia* meaning "angle." The following are polygons:



59. Triangle, (△). A triangle is a polygon of three sides. Triangles are sometimes called *trigons*. *Tri* means "three," as in *tricycle* or *triplane*.

60. Perimeter of a Polygon. The length of the broken line which forms a polygon is called its *perimeter*. It is equal to the sum of the lengths of the sides. *Peri* means "around" and *metron* means "measure." The figures *ABCDE* and *ABC* are polygons; *AB*, *BC*, *CD*, etc., are called *sides;* *A*, *B*, *C*, etc., are called *vertices;* $AB + BC + CA$ is the perimeter of the triangle *ABC*. What is the perimeter of *ABCDE*?



61. Kinds of Triangles. A triangle is called,

equilateral if three sides are equal;

isosceles if two sides are equal;

scalene if no sides are equal;

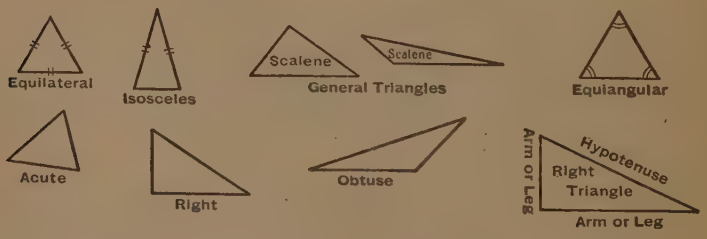
equiangular if three angles are equal;

acute if three angles are acute;

right if one angle is right;

obtuse if one angle is obtuse.

62. Named Parts of a Triangle. Any side of a triangle may be called a *base*, while the angle opposite that base is

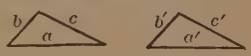


called the *vertex angle* of the triangle. In a right triangle, the side opposite the right angle is called the *hypotenuse* and the two sides of the right angle are called either *legs* or *arms*. In an isosceles triangle, the equal sides are called *legs*, the

third side is called the *base;* the angles adjacent to the base are called *base angles.*

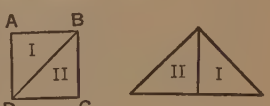
63. The General Triangle. The most general triangle is the scalene triangle and when the word *triangle* is used with no qualifying adjectives, a *scalene* triangle is meant. For example, on page 69, we prove that the sum of the angles of a triangle equals a straight angle. The proof does not depend upon the equality of two or three of its sides nor upon the size of any of its angles, hence a scalene triangle is used as the figure upon which the proof is based.

64. Congruent Figures. Two closed figures which can be made to coincide are said to be *congruent.* For example, the two triangles *abc* and $a'b'c'$ are congruent if when a is placed upon a', b falls along b' and c falls along c'.



It should be noted that when two open figures, such as two line segments or two angles, are made to coincide they are said to be *equal,* while two closed figures which coincide are called *congruent.* Congruent figures are often said to possess the same size and shape.

Two closed figures which have the same size are not necessarily congruent. For example, the square and the large triangle have the same size, but not the same shape; they cannot be made to coincide and therefore are not congruent.

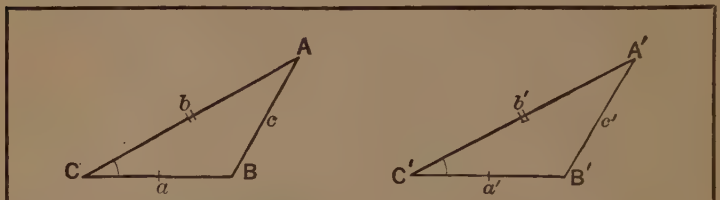


The symbol for congruence is $\cong$. It is composed of two symbols; $=$ means "having equal size," and $\sim$ means "having the same shape." $\triangle ADB \cong \triangle CBD$ is read, "triangle ADB is congruent to triangle CBD."

NOTE: Propositions 1 and 2 may be assumed, but their use in exercises should be emphasized to make clear the meaning of the theorems and their importance as tools. The fundamental character of superposition precludes the discarding of these proofs.

Proposition 1 (C N)

65. Theorem. *Triangles with two sides and the included angle of one equal to the respective parts of another are congruent.* (*s.a.s.* = *s.a.s.*)



Given: $\triangle ABC$ and $A'B'C'$, $b = b'$, $a = a'$, $\angle C = \angle C'$.

To prove: $\triangle ABC \cong \triangle A'B'C'$.

HINT: What are congruent triangles?

Proof: STATEMENTS	FACTS
1. $b = b'$.	1. Given.
2. b can be made to coincide with b'.	2. Two equal line segments can be made to coincide by matching their respective end points. (§ 24.)
3. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that b will coincide with its equal b'. (Point A upon point A' and point C upon point C'.)	3. A geometric figure may be moved about without change of shape or size. (*Prin.* 37.)
4. a will extend along a'.	4. $\angle C = \angle C'$, given.
5. Point B will fall upon point B'.	5. $a = a'$, given.
6. c will coincide with c'.	6. One straight line and only one can be drawn from one point to another. (*Prin.* 13.)
7. $\triangle ABC \cong \triangle A'B'C'$.	7. Definition. (§ 64.)

66. The Method of Superposition. The method of proof given for Proposition 1 is called a proof by *superposition.* The word *superposition* means "placing upon." One triangle was placed upon the other triangle. This method is seldom employed in the proofs of our course; it is used primarily to establish the truths of the elementary or basic theorems of certain families. In a proof of this kind, we begin by placing one end of a side of the first triangle upon one end of a side of the second triangle making the two lines extend in the same direction from the matched points. If the two lines are given equal, one may be placed upon the other, end upon end, for equal lines may be made to coincide. The direction taken by each of the remaining lines (determined by the angles) and the lengths of those lines locate the positions of the unmatched points of the two triangles.

Exercises

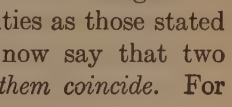
1. In the figure for Proposition 1, if $b = b'$, $a = a'$ and $\angle C \neq \angle C'$, draw two sketches of what may happen after b is placed upon b'. (The symbol $\neq$ is read, "is not equal to.")

2. In like manner, draw two sketches showing the conditions when $b = b'$, $a \neq a'$ and $\angle C = \angle C'$.

3. In the figure for Proposition 1, if $b = b'$, $c = c'$, and $\angle A = \angle A'$, what is the relation of the triangles?

67. Since Proposition 1 has been established by proof, it may be used as a *fact* to prove that any two triangles are congruent if they possess the same equalities as those stated in the proposition. That is, we may now say that two triangles are congruent *without making them coincide.* For example:

If $AO = OC$ and $DO = OB$ when AC intersects DB at O, we may establish the congruence of $\triangle AOD$ and $\triangle BOC$ by stating:



In $\triangle AOD$ and BOC,	
1. $AO = OC$.	1. Given.
2. $DO = OB$.	2 Given.
3. $\angle 1 = \angle 2$.	3. Vertical angles are equal. (*Prin.* 27.)
4. $\triangle AOD \cong \triangle BOC$.	4. Proposition 1. (*s.a.s.* = *s.a.s.*)

68. Corollary. A corollary is a statement whose proof is the consequence of some other truth to which it is related.

69. Corollary of Proposition 1. *Right triangles with their legs respectively equal are congruent.* (*l.l.* = *l.l.*)

HINT: As in the example above, establish this truth by showing that *s.a.s.* = *s.a.s.*

70. The Corollary Method. The method used to prove the example of § 67 and the corollary of § 69 may be called the *Corollary Method.* Here a new truth is proved by showing that the given hypothesis may be reduced to the hypothesis of a previously proved theorem. This method is used many times in our work; it is especially important in proving originals or exercises.

Sight Exercises. Corollary Method

Write or state the proofs of the following exercises in the form shown in § 67:

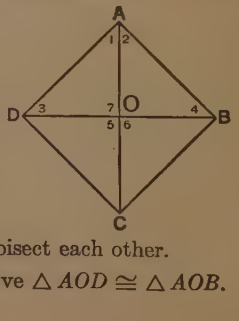
1. If $AD = AB$, $\angle 3 = \angle 4$, and $DO = OB$, prove $\triangle AOD \cong \triangle AOB$.

2. If $DO = OB$ and $\angle 5 = \angle 6$, prove $\triangle DOC \cong \triangle COB$.

3. If $AD = AB$ and $\angle 1 = \angle 2$, prove $\triangle AOD \cong \triangle AOB$.

4. Prove $\triangle AOD \cong \triangle AOB$, if $DO = OB$ and $AO \perp DB$.

5. Prove $\triangle AOD \cong \triangle COB$ if $OA = OC$ and $OD = OB$.

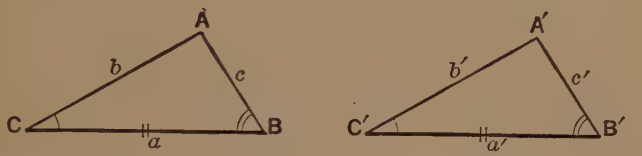


6. Prove $\triangle COB \cong \triangle AOD$ if AC and DB bisect each other.

7. If $\triangle ABD$ is isosceles and $\angle 1 = \angle 2$, prove $\triangle AOD \cong \triangle AOB$.

Proposition 2 (C N)

71. Theorem. *Triangles with two angles and the included side of one equal to the respective parts of another are congruent.* (*a.s.a.* = *a.s.a.*)



Given: $\triangle ABC$ and $A'B'C'$, $\angle C = \angle C'$, $\angle B = \angle B'$, $a = a'$.

To prove: $\triangle ABC \cong \triangle A'B'C'$.

HINT: Use the Method of Superposition. (See § 66.)

Proof: STATEMENTS	FACTS
1. $a = a'$.	1. Given.
2. a can be made to coincide with a'.	2. § 24.
3. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that a will coincide with its equal a', point B upon B', point C upon point C'.	3. A geometric figure may be moved without change of size or shape. (*Prin.* 37.)
4. b will extend along b'.	4. $\angle C = \angle C'$. Given.
5. c will extend along c'.	5. $\angle B = \angle B'$. Given.
6. Point A will fall upon point A'.	6. Two straight lines can intersect in only one point. (*Prin.* 20.)
7. $\triangle ABC \cong \triangle A'B'C'$.	7. Figures which coincide in all their parts are congruent. (Definition.)

72. Corollary. *Right triangles with a leg and two adjacent angles of one equal to the respective parts of another are congruent.* (*l.a.* = *l.a.*)

HINT: Use the Corollary Method. (See § 70.)

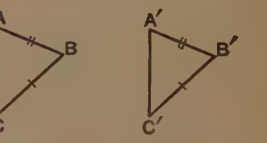
Exercises

1. In the figure for Proposition 2, if $a = a'$, $\angle B = \angle B'$, and $\angle C \neq \angle C'$, show by two sketches what may happen when a is placed upon a'.

2. In like manner, show what may happen when $a \neq a'$, $\angle B = \angle B'$, and $\angle C = \angle C'$.

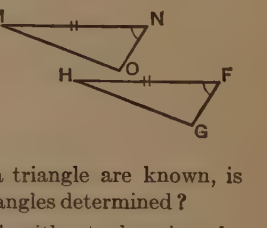
Oral Exercises

1. In $\triangle ABC$ and $A'B'C'$, if $AB = A'B'$ and $BC = B'C'$, what other pair of parts must be equal in order to prove the triangles congruent?



2. In order to prove $\triangle XYZ \cong \triangle X'Y'Z'$, what other pair of parts must be equal besides $\angle Z = \angle Z'$ and $\angle X = \angle X'$?

3. In triangles MNO and HFG, if $MN = HF$ and $\angle N = \angle F$, what other pair of angles (or of sides) must be equal to make $\triangle MNO \cong \triangle HFG$?



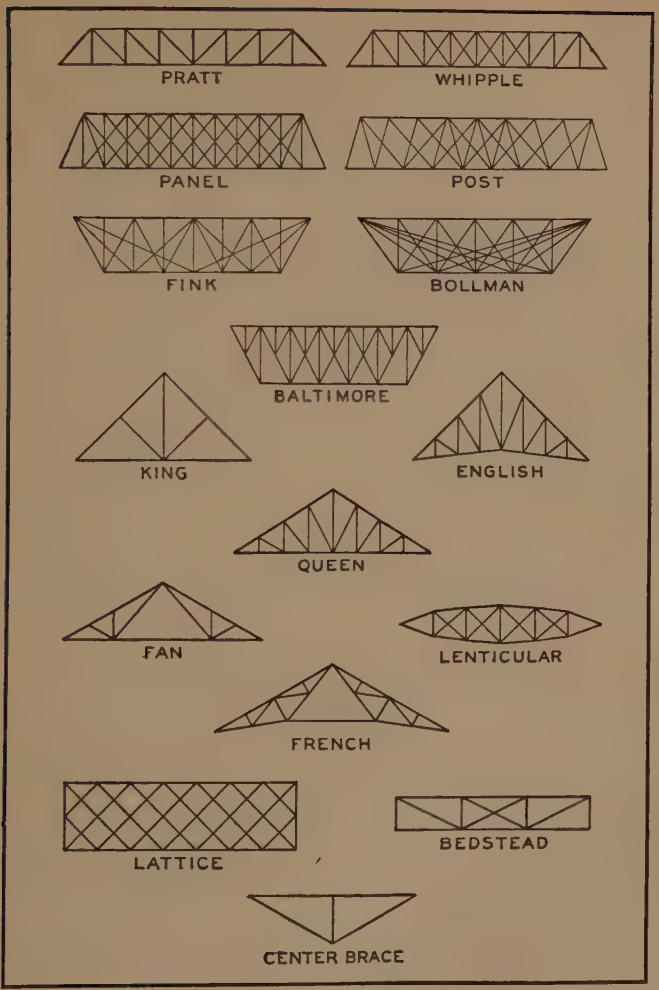
4. If two sides and the included angle of a triangle are known, is the third side determined? Are the other two angles determined?

5. May the shape of a triangle be changed without changing the lengths of the sides? (See the illustrations of trusses, p. 47.)

6. May any of the parts of a triangle be changed without changing the shape and the size of the triangle?

7. Is a triangle determined, that is, are all the parts fixed, if two angles and the included side are known? if two sides and the included angle are known?

8. In the figures for Proposition 2, state which of the following conditions make the triangles congruent: (*a*) $b = b'$, $c = c'$, $\angle B = \angle B'$; (*b*) $c = c'$, $a = a'$, $\angle B = B'$; (*c*) $\angle A = \angle A'$, $\angle B = \angle B'$, $c = c'$; (*d*) $b = b'$, $c = c'$, $\angle C = \angle C'$; (*e*) $\angle A = \angle A'$, $\angle B = \angle B'$, $a = a'$; (*f*) $\angle B = \angle B'$, $\angle C = \angle C'$, $a = a'$.



TRUSSES

A truss is a combination of beams, rods, bars, etc., so as to form a rigid framework. The form of a truss should be that of a triangle or combination of triangles because a triangle is the only polygon whose shape cannot alter without changing the length of a side. Trusses are used in buildings and bridges where a load must be supported over a wide open space.

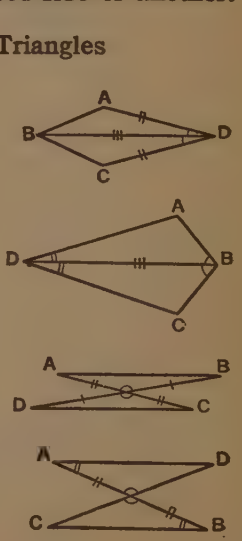
73. Tools of Plane Geometry. In everyday life some tools are used so frequently by certain workmen that they are known as the tools of a particular trade. The hammer, plane, and saw belong to the carpenter; the wrench, screw-driver, and jack belong to the auto-mechanic; the plow, tractor, and harvester belong to the farmer. In geometry, definitions, principles, and theorems (previously proved) are tools used to establish new facts. Some of these are used so frequently that they stand out as the *tools of geometry*. From time to time attention will be called to them under appropriate headings.

74. Tool I. Triangles may be proved congruent by showing that two sides and the included angle of one are equal respectively to two sides and the included angle of another.

75. Tool II. Triangles may be proved congruent by showing that two angles and the included side of one are equal respectively to two angles and the included side of another.

Sight Exercises. Congruent Triangles

In the following exercises use the Corollary Method of proof with the help of Tool I or Tool II:



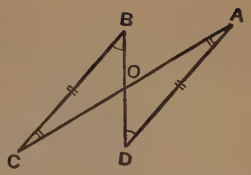
1. In the annexed figure, if AD and CD are equal and DB bisects their included angle, it divides $ABCD$ into two congruent triangles.

2. In the annexed figure, if DB bisects the angles D and B, it divides $ABCD$ into two congruent triangles.

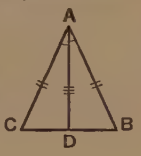
3. If two lines bisect each other, lines joining their extremities (as in the figure) form congruent triangles.

4. In the annexed figure, if AB is bisected by CD and if AB forms equal angles with AD and BC, prove two triangles congruent.

5. In the annexed figure, if AD and BC are equal and if the angles they form with the intersecting lines are equal resepectively, two congruent triangles are formed.

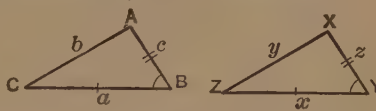


6. The bisector of the angle at the vertex of an isosceles triangle forms two congruent triangles.



76. Corresponding Parts. When two congruent figures are superposed, the points, lines, and angles of the one which coincide respectively with the points, lines, and angles of the other are called *corresponding parts.* Any pair of the known equal parts are corresponding and the pair of parts which lie opposite them are also corresponding. For example:

In the annexed figure, if we know that the triangles are congruent because $c = z$, $a = x$, and $\angle B = \angle Y$,



then these known parts occur in *pairs of corresponding parts.* The remaining three pairs of parts are matched into three pairs of corresponding parts as they *lie opposite the known equal* pairs of *parts,* that is, $\angle C$ and $\angle Z$, $\angle A$ and $\angle X$, and b and y are also corresponding parts.

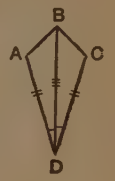
77. *Corresponding parts of congruent figures are equal.* (*Prin.* 38.)

78. Tool III. Two lines or two angles may be proved equal by showing that they are corresponding parts of congruent figures.

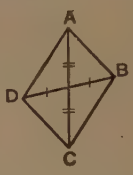
First, fix in mind the two lines or the two angles; second, select two figures (usually triangles) of which they are parts; third, prove these figures congruent; fourth, state the required equality of corresponding parts.

Exercises. Corresponding Parts (Use Tool III.)

1. In the annexed figure, if AD and CD are equal and DB bisects their included angle, then AB equals CB.



2. In the same figure, if DB bisects the angles D and B, then $AD = DC$, and $AB = BC$.



3. If two lines bisect each other, the opposite lines joining their extremities are equal, in pairs.

4. In the figure, if $AO = OB$ and $\angle A = \angle B$, then $AD = CB$, $\angle D = \angle C$, and $CO = OD$.

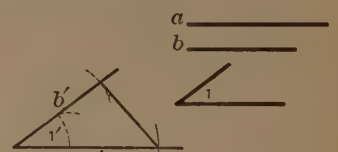
Drawing Exercises. Triangles

1. Draw a triangle when two sides and the included angle are given.

Given: a, $\angle 1$, and b.

To find the triangle.

(Let the pupil supply the description from the annexed figure.)

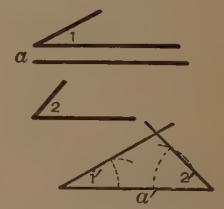


2. Draw a triangle when two angles and the included side are given.

Given: $\angle 1$, a, and $\angle 2$.

To find the triangle.

(Let the pupil supply the description from the annexed figure.)

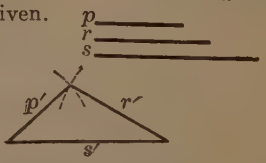


3. Draw a triangle when its sides are given.

Given: p, r, and s.

To find the triangle.

(Let the pupil supply the description from the annexed figure.)



4. Draw an isosceles triangle. Describe the method used.

5. Draw an equilateral traingle. Describe the method used.

Courtesy Delaware River Bridge Joint Com.

DELAWARE RIVER BRIDGE, BETWEEN CAMDEN AND PHILADELPHIA

Geometry played its part in making it possible for the two sections to meet in the center. The height of the span above the surface of the river is greatest at the center. Why does this give more strength than if the span were parallel to the surface of the water?

Oral Exercises

1. What is an isosceles triangle?

2. Define an equilateral triangle.

3. Is an equilateral triangle isosceles?

4. If an angle of 70° is bisected, into how many parts is it divided? What is the size of each part?

5. How long is the bisector of an angle?

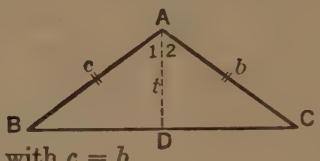
6. What is meant by corresponding parts of congruent figures?

7. How do you match the parts of congruent figures into pairs of corresponding parts?

8. Are two right triangles congruent if their legs are equal? if their legs are respectively equal? Draw figures in each case and give the fact for each answer.

Proposition 3 (C N)

79. Theorem. *Base angles of an isosceles triangle are equal.*



Given: $\triangle ABC$ with $c = b$.

To prove: $\angle B = \angle C$.

HINT: Draw the bisector of $\angle A$ and prove that the triangles thus formed are congruent.

Proof:

STATEMENTS	FACTS
1. Draw t bisecting $\angle A$ and meeting BC at some point D.	1. An angle may be bisected. (*Prin.* 25.)
In $\triangle ABD$ and $\triangle ACD$,	
2. $c = b$.	2. Given.
3. $\angle 1 = \angle 2$.	3. By construction.
4. $t = t$.	4. Identical. (*Prin.* 1.)
5. $\triangle ABD \cong \triangle ACD$.	5. *s.a.s.* = *s.a.s.*
6. $\angle B = \angle C$.	6. Corresponding parts of congruent figures are equal. (*Prin.* 38.)

80. Corollary 1. *Angles opposite equal sides in the same triangle are equal.*

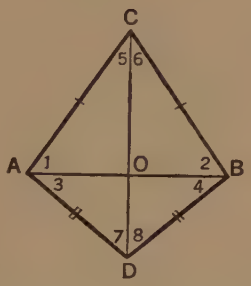
81. Corollary 2. *An equilateral triangle is equiangular.*

HINT: Show the three angles are equal by § 80.

82. Tool IV. Angles may be proved equal by showing that they lie opposite equal sides *in the same triangle.*

Proposition 4

83. Theorem. *The line joining the vertices of two isosceles triangles on the same base bisects the angles at those vertices.*



Given: $\triangle CAB$ and DAB, $AC = BC$, $AD = BD$.

To prove: $\angle 5 = \angle 6$ and $\angle 7 = \angle 8$.

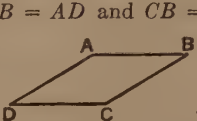
HINT: What condition frequently makes angles equal?

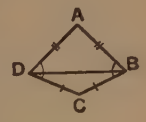
Proof:

STATEMENTS	FACTS
In $\triangle ACD$ and $\triangle BCD$,	
1. $AC = BC$ and $AD = BD$.	1. Given.
2. $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$.	2. § 80.
3. $\angle 1 + \angle 3 = \angle 2 + \angle 4$.	3. *Prin.* 5.
4. $\triangle ACD \cong \triangle BCD$.	4. *s.a.s.* = *s.a.s.*
5. $\angle 5 = \angle 6$ and $\angle 7 = \angle 8$.	5. *Prin.* 38.

Exercises. Equal Angles (Use Tool IV.)

1. In the annexed figure, if $AB = AD$ and $CB = CD$, prove that $\angle B = \angle D$.



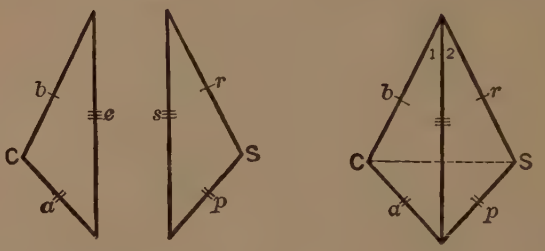


2. In the figure, if $AD = BC = AB = DC$, prove that $\angle A = \angle C$ and $\angle B = \angle D$.

3. In the figure, if $AB = AC$, $AE = EC$, and $AD = DB$, prove $\angle 1 = \angle 2$.

Proposition 5 (C N)

84. Theorem. *Triangles with their sides respectively equal are congruent.* (*s.s.s.* = *s.s.s.*)



Given: $\triangle abc$ and prs, $a = p$, $b = r$, $c = s$.

To prove: $\triangle abc \cong \triangle prs$.

HINT: The proof follows the Corollary Method. Use the preceding theorem (§ 83) as a tool to show that a pair of angles are equal and so establish the conditions for *s.a.s.* = *s.a.s.*

Proof: STATEMENTS	FACTS
1. $c = s$.	1. Given.
2. c can be made to coincide with s.	2. § 24.
3. Place $\triangle abc$ so that c falls upon its equal s.	3. *Prin.* 37.
4. Draw CS.	4. *Prin.* 13.
5. $a = p$.	5. Given.
6. $b = r$.	6. Given.
7. $\angle 1 = \angle 2$.	7. § 83.
8. $c = s$.	8. Given.
By Statements 6, 7, and 8,	
9. $\triangle abc \cong \triangle prs$.	9. *s.a.s.* = *s.a.s.*

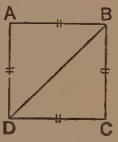
NOTE: Right triangles with the *hypotenuse* and *legs* of one equal to the respective parts of another are congruent.

85. Tool V. Triangles may be proved congruent by showing that three sides of one are equal respectively to the three sides of another.

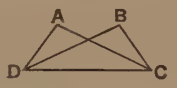
Exercises. Congruent Triangles

1. Two equilateral triangles are congruent if a side of the one equals a side of the other.

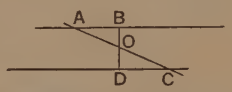
2. In the annexed figure, BD divides the square into two congruent triangles.



3. In the annexed figure, $AC = BD$ and $AD = BC$, prove $\triangle ADC \cong \triangle BCD$.



4. In the figure, if $AB = CD$ and AC and BD bisect each other at O, prove that $\triangle OBA \cong \triangle ODC$. (Two methods.)



5. State, in full, three ways by which two triangles are proved congruent.

86. Auxiliary Construction Lines. In the proofs of Propositions 3 and 5 (§§ 79 and 84), lines not given or defined by the theorems were drawn to aid in the proofs. Such lines are called *auxiliary construction lines*. They should always be represented as *dotted* lines. In the drawing of such lines care should be taken to make them *answer only two independent conditions*, one of which is that *it must pass through a given point.*

For example: a line may be drawn

a) through two *given points*,
b) through a *given point* and *parallel* to a given line,
c) through a *given point* and *perpendicular* to a given line,
d) through a *given point* and *equal* to a given line, or
e) *bisecting* a given angle. (Here the two conditions are; first, it passes through *the vertex;* second, it divides the angle into *two equal angles*.)

Although a line may and does answer many conditions after it is drawn, only two of those conditions may be assigned to the line when it is first constructed. For example, if ABC is an equilateral triangle and AD is drawn bisecting $\angle A$, then AD is restricted to these two conditions:

a) it passes through point A, and

b) it makes $\angle 1 = \angle 2$.

But after AD is drawn, it may be shown to answer many more conditions, such as:

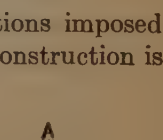
(1) it makes $\angle 3 = \angle 4 =$ right angle;
(2) it makes $CD = DB$;
(3) it divides $\triangle ABC$ into two congruent triangles;
(4) it equals $\sqrt{3} \times CD$;
(5) it passes through innumerable points between A and D.

87. The Fact Called Construction. When an auxiliary construction line is drawn to aid in a proof, the two conditions imposed upon the line may be used among the *statements of the proof* when they are supported by the *Fact called Construction*. Observe that this has already been done for Fact 3 of § 79. *If any other property of the line is used in a proof, it must be supported in some other manner* (which is usually the statement of some previously proved theorem).

Exercises. Conditions of Construction

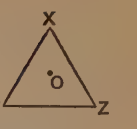
In the following exercises state the conditions imposed upon the line to be drawn and whether the construction is possible:

1. Draw a line $AD \perp BC$.
2. Draw a line AD bisecting BC.
3. Draw a line AD bisecting $\angle A$.
4. Draw the perpendicular bisector of BC from A.



5. Draw a line bisecting $\angle A$ and $\perp BC$.

6. Join the mid-point of YZ with O and produce it through X.



88. Finding Equalities for Congruence. In selecting the equalities necessary to prove triangles congruent:

(1) make the fullest use possible of the hypothesis; (this will usually give one or more equalities directly.)

(2) study the figure to find relations that give all the other equalities;

(3) mark equal parts with like checks.

Remember especially that:

a) intersecting lines have equal vertical angles:

b) perpendicular lines have equal right angles;

c) an isosceles triangle has equal sides and that its base angles may be shown to be equal;

d) the mid-point or the bisector of a line makes equal line segments;

e) the bisector of an angle makes equal angles;

f) the common side or the common angle of two triangles may be used in an equality.

Oral Exercises

In each of the following exercises, fix in mind the two given equalities. Select the two triangles of which they are parts. What third pair of parts is equal in those triangles? Why? Quote the theorem which states why the triangles are congruent.

1. If $OS = OR$ and $\angle 2 = \angle 5$;

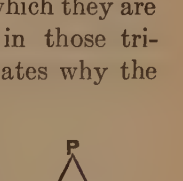
2. If $PS = PR$ and $\angle 2 = \angle 5$;

3. If $OS = OR$ and $OX = OY$;

4. If $OS = OR$ and $ST = TR$;

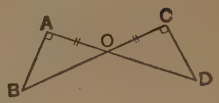
5. If $PS = PR$ and $PY = PX$;

6. If $SX = RY$ and $RX = SY$.

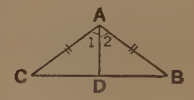


Sight Exercises. Congruent Triangles

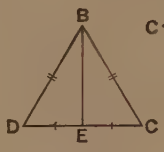
1. **Given:** $AB \perp AD$; $CD \perp BC$; $AO = OC$.
   To prove: $\triangle AOB \cong \triangle COD$.



2. **Given:** $AC = AB$; $\angle 1 = \angle 2$.
   To prove: $\triangle ACD \cong \triangle ABD$.



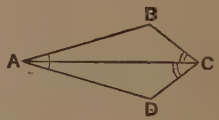
3. **Given:** $BD = BC$; $DE = EC$.
   To prove: $\triangle BDE \cong \triangle BCE$.



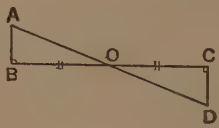
4. **Given:** $PT \perp RS$; $ST = TR$.
   To prove: $\triangle PST \cong \triangle PRT$.

5. **Given:** $CH \perp DE$; $\angle 1 = \angle 2$.
   To prove: $\triangle CEH \cong \triangle CDH$.

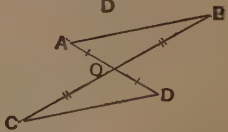
6. **Given:** AC is the bisector of $\angle A$ and $\angle C$.
   To prove: $\triangle ADC \cong \triangle ABC$.



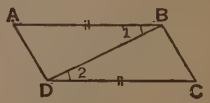
7. **Given:** $AB \perp BC$; $CD \perp BC$; O is the mid-point of BC.
   To prove: $\triangle ABO \cong \triangle CDO$.



8. **Given:** AD and BC bisect each other.
   To prove: $\triangle ABO \cong \triangle CDO$.



9. **Given:** $AB = CD$; $\angle 1 = \angle 2$.
   To prove: $\triangle ABD \cong \triangle BCD$.



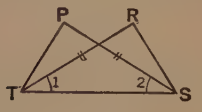
10. **Given:** $ABCDE$ is equilateral; $\angle E = \angle B$.
    To prove: $\triangle ADE \cong \triangle ABC$.

11. **Given:** $AO = CO$;
$BO = DO$.

To prove: $\triangle ABO \cong \triangle CDO$.

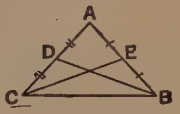
12. **Given:** $PS = RT$;
$\angle 1 = \angle 2$.

To prove: $\triangle PST \cong \triangle RST$.



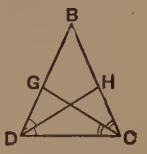
13. **Given:** $AC = AB$;
D and E are mid-points of AC and AB.

To prove: $\triangle ACE \cong \triangle ABD$, and
$\triangle BCD \cong \triangle BCE$.



14. **Given:** $BD = BC$;
CG and DH are the bisectors of $\angle C$ and $\angle D$.

To prove: $\triangle BDH \cong \triangle BCG$ and
$\triangle CDG \cong \triangle CDH$.

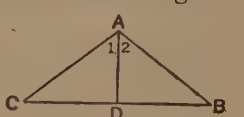


15. A line from the vertex of an isosceles triangle to the mid-point of the base forms congruent triangles.

Exercises. Equal Lines and Equal Angles

With the aid of Tools III and IV prove the following exercises:

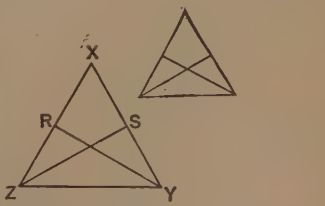
1. $\triangle ABC$ is isosceles and $CD = DB$; prove $\angle 1 = \angle 2$.



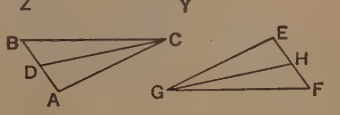
2. A line which bisects the angle at the vertex of an isosceles triangle bisects the base.

3. The bisectors of the base angles of an isosceles triangle are equal.

4. In the annexed figure, if $XZ = XY$, $XR = RZ$, and $XS = SY$, prove $ZS = YR$.

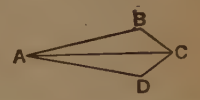


5. Triangles ABC and EFG are congruent and $AD = DB$ and $FH = HE$, then $CD = GH$.



6. The bisectors of corresponding angles of congruent triangles are equal.

7. Using the annexed figure, describe how a surveyor may measure the distance MN by finding PR.

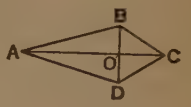


8. If $AB = AD$ and $CB = CD$, then AC bisects the angles included by the equal sides.

89. Two-step Proofs. When it is impossible to prove the congruence of a pair of triangles by using only the given conditions, try to prove first the congruence of some other pair of triangles in the figure, whose corresponding parts will aid in establishing the congruence of the required pair.

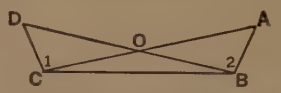
Exercises. Two-step Proofs

1. In the annexed figure, if $AB = AD$ and $BC = CD$, prove $\triangle ABO \cong \triangle AOD$.



HINT: First prove $\triangle ABC \cong \triangle ADC$.

2. In the figure, if $AC = BD$ and $AB = CD$, prove $\triangle AOB \cong \triangle DOC$.

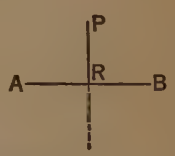


HINT: Prove $\angle 1 = \angle 2$, by the Axiom of Subtraction.

3. In the figure, if $AB = AD$ and $BO = OD$, prove $\triangle BOC \cong \triangle DOC$.

Oral Exercises

1. Given $PR \perp AB$. Required to draw a second line through P and perpendicular to AB. Describe the position of such a line. (*Prin.* 15.)



2. May two lines be so placed that one coincides with the other?

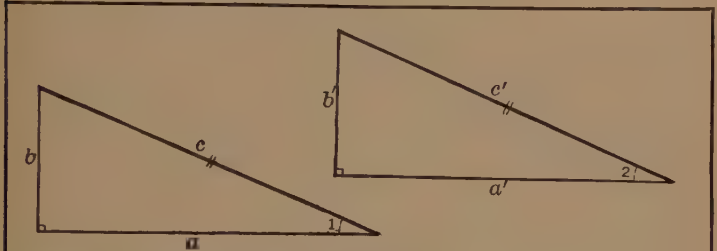
3. Which two sides of a right triangle are perpendicular?

4. What kind of an angle lies opposite the hypotenuse of a right triangle?

5. How many angles of a right triangle are adjacent to the hypotenuse?

Proposition 6 (c)

90. Theorem. *Right triangles with the hypotenuse and an adjacent angle of one equal to the respective parts of another are congruent.* (*hy.a.* = *hy.a.*)



Given: $\triangle abc$ and $a'b'c'$, $b \perp a$, $b' \perp a'$, $\angle 1 = \angle 2$, $c = c'$.

To prove: Rt. $\triangle abc \cong$ rt. $\triangle a'b'c'$.

HINT: Use the Method of Superposition. (See § 66.)

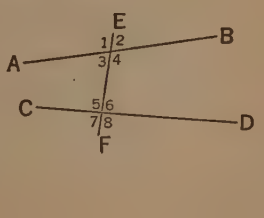
Proof:

STATEMENTS	FACTS
1. $c = c'$.	1. Given.
2. c can be made to coincide with c'.	2. § 24.
3. Place rt. $\triangle abc$ upon rt. $\triangle a'b'c'$ so that c falls upon c'.	3. *Prin.* 37.
4. a extends along a'.	4. $\angle 1 = \angle 2$. Given.
5. $b \perp a$ and $b' \perp a'$.	5. Given.
6. b and b' coincide.	6. *Prin.* 15.
7. rt. $\triangle abc \cong$ rt. $\triangle a'b'c'$.	7. Figures which coincide in all their parts are congruent.

ANGLES FORMED BY A TRANSVERSAL

91. Transversal. A line intersecting each of two or more lines is called a *transversal* of those lines.

EF is a transversal of *AB* and *CD*.



Oral Exercises (Fig. of § 91)

1. What kind of angles are 2 and 3? How many pairs of such angles are formed? Name the pairs.

2. What kind of angles are 1 and 2? Name the other pairs of such angles?

3. How many pairs of adjacent angles are found in the figure? Name them.

92. Interior Angle. An angle formed by either of two lines and the part of the transversal which is cut off by them is called an *interior angle.* In the figure of § 91, $\angle 3$, $\angle 4$, $\angle 5$, and $\angle 6$ are interior angles.

93. Exterior Angle. An angle formed by either of two lines and a part of the transversal which is not cut off by them is called an *exterior angle.* In the figure of § 91, $\angle 1$, $\angle 2$, $\angle 7$, and $\angle 8$ are exterior angles.

94. Alternate Interior Angles. Two interior angles on opposite sides of the transversal and not adjacent are called *alternate interior angles.*

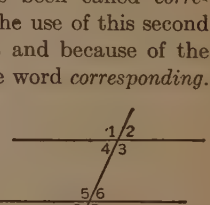
95. Alternate Exterior Angles. Two exterior angles on opposite sides of the transversal and not adjacent are called *alternate exterior angles.*

96. Exterior-Interior Angles. An exterior angle and an interior angle on the same side of the transversal and not adjacent are called *exterior-interior angles.*

NOTE: Exterior-interior angles have sometimes been called *corresponding angles.* Many teachers prefer to avoid the use of this second name because it is not as descriptive as the first and because of the confusion which may arise due to other uses of the word *corresponding.*

In the adjoining figure,

4 and 6 are alternate interior angles;
1 and 7 are alternate exterior angles;
1 and 5 are exterior-interior angles.



Oral Exercises

1. Name $\angle 2$ and $\angle 4$; $\angle 1$ and $\angle 2$; $\angle 4$ and $\angle 6$.

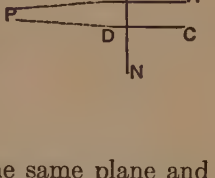
2. Name $\angle 3$ and $\angle 5$; $\angle 2$ and $\angle 8$; $\angle 2$ and $\angle 6$.

3. Name $\angle 4$ and $\angle 8$; $\angle 5$ and $\angle 6$; $\angle 5$ and $\angle 7$.

4. Describe $\angle 2$ and $\angle 7$ using the word exterior.

5. Describe $\angle 4$ and $\angle 5$ using the word interior.

6. In the annexed figure, $AB \perp MN$ and $CD \perp MN$. Suppose AB and CD when produced meet at some point such as P. How many perpendiculars would then be drawn from P to MN? Is it possible to have two perpendiculars from a point to a line? (*Prin.* 15.)



PARALLEL LINES

97. Parallel Lines. Lines that lie in the same plane and that cannot meet however far produced are called *parallel lines*.

If two lines are parallel, any segment of one is parallel to any segment of the other. Lines which are not parallel will meet if produced far enough.

98. Corollary (c). *Two lines perpendicular to the same straight line are parallel.*

For if they were not parallel and met at some point, there would be two perpendiculars from that point to the line. This is impossible because of Principle 15.

Oral Exercises

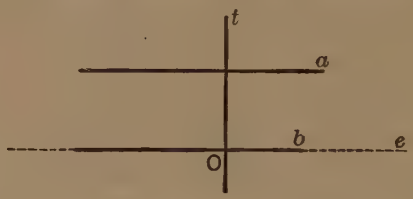
1. If two telegraph wires cross at a street corner, one above the other, and would never meet, are they examples of parallel lines? Why?

2. Given $AB \parallel CD$ through P. Required to draw a second line through P and parallel to CD. Describe the position of such a line. Principle 14 states that, "Through a point one line and only one can be drawn parallel to a given line."

3. If two lines e and b coincide and e is perpendicular to a line t, what is the relation of line b to line t?

Proposition 7 (c n)

99. Theorem. *A line perpendicular to one of two parallel lines is perpendicular to the other.*



Given: $a \parallel b$, $t \perp a$ and meeting b at O.

To prove: $t \perp b$.

HINT: (See Nos. 2 and 3 of the preceding exercises.)

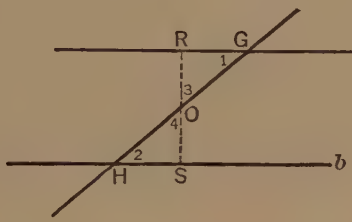
Proof: STATEMENTS	FACTS
1. Suppose e be drawn perpendicular to t through O.	1. A perpendicular may be drawn to a given line through a given point. (*Prin.* 15.)
2. $a \perp t$.	2. Given.
3. Then e is parallel to a.	3. Two lines in the same plane perpendicular to the same line are parallel. (*Prin.* 17.)
4. But b is parallel to a.	4. Given.
5. Since e and b are both parallel to a through O, e and b coincide.	5. Through a point one line and only one can be drawn parallel to a given line. (*Prin.* 14.)
6. $b \perp t$.	6. Statement 1.

100. Coincidence Method. The proof of Proposition 7 follows what is called the *Method of Coincidence.* Here the given line b is shown to coincide with another line e which

answers the required condition (that is, of being perpendicular to t). This method is one of three *indirect forms of proof* which are so called because they do not build the conclusion directly.

Proposition 8 (C N)

101. Theorem. *Alternate interior angles of parallel lines are equal.*

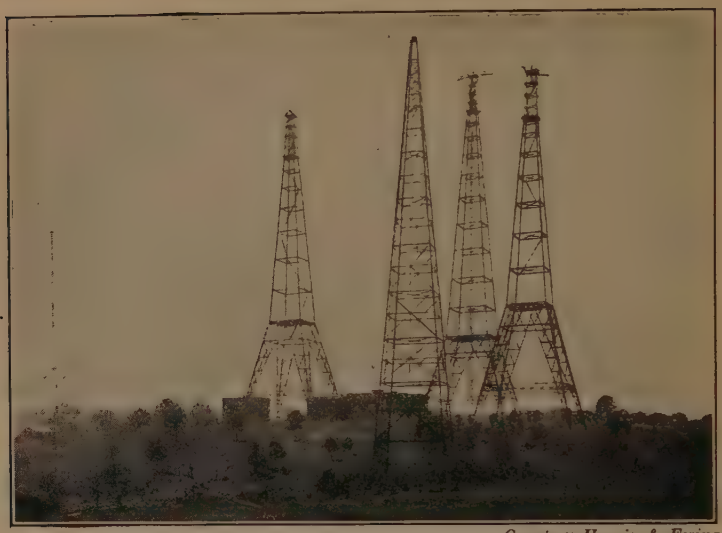


Given: $a \parallel b$ with GH making alt. int. $\measuredangle$ 1 and 2.

To prove: $\angle 1 = \angle 2$.

HINT: Form triangles in which $\angle 1$ and $\angle 2$ are respective parts. What is the advantage of drawing a perpendicular through O?

Proof: STATEMENTS	FACTS
1. Let O be the mid-point of GH.	1. *Prin.* 12.
2. Let ROS be drawn $\perp b$.	2. *Prin.* 15.
3. $a \parallel b$.	3. Given.
4. $OR \perp a$.	4. § 99.
In rt. $\triangle ORG$ and OHS,	
5. $OG = OH$.	5. Cons.
6. $\angle 3 = \angle 4$.	6. Vert. $\measuredangle$ are equal.
7. rt. $\triangle ORG \cong$ rt. $\triangle OHS$.	7. § 90.
8. $\angle 1 = \angle 2$.	8. § 77.

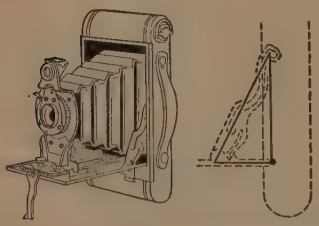


Courtesy Harris & Ewing

THE NAVAL RADIO STATION, ARLINGTON, VA.

The whole structure of each of these towers is composed of triangle upon triangle. The tallest tower is 600 ft.; the others are each 450 ft. in height.

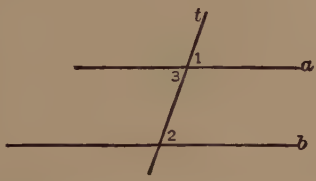
Why is it unsafe to stand upon a stepladder if the braces are not fully extended?



Note the locking brace on the camera which completes a triangle.

Many familiar objects owe their strength and firmness to the rigidity of the triangle. Why is the triangle a rigid figure?

102. Corollary (c n). *Exterior-interior angles of parallel lines are equal.*



Given: $a \parallel b$ with t making ext.-int. $\measuredangle$ 1 and 2.

To prove: $\angle 1 = \angle 2$.

Proof:

STATEMENTS	FACTS
1. $\angle 1 = \angle 3$.	1. *Prin.* 27.
2. $a \parallel b$.	2. Given.
3. $\angle 3 = \angle 2$.	3. § 101.
4. $\angle 1 = \angle 3 = \angle 2$.	4. Sub. (*Prin.* 3.)

103. The Substitution Method. The method used to prove the preceding corollary may be called the *Substitution Method.* It consists of three principal steps.

General directions for use of the Substitution Method are as follows:

(1) State a fact which includes one or more quantities of the conclusion and which conforms with the conclusion. as $\angle 1 = \angle 3$ in § 102;

(2) Compare this preliminary statement with the conclusion and find the relation between any new quantities which are introduced and other quantities of the conclusion, as $\angle 3 = \angle 2$;

(3) Use one or more substitutions to obtain the conclusion, as $\angle 1 = \angle 3 = \angle 2$.

It is seldom possible to limit such a proof to only three statements; other statements are usually necessary in order to form a basis for one of the principal steps, as $a \parallel b$ is the basis which permits the statement $\angle 3 = \angle 2$ to be made.

The Substitution Method is of especial advantage when a conclusion is an arithmetic or an algebraic relation.

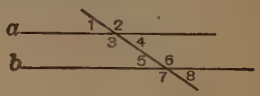
104. Corollary (c n). *Interior angles on the same side of the transversal of parallel lines are supplementary.*

HINT: Use the Substitution Method.

105. Tool VI. Two angles may be proved equal by showing that they are $\left\{\begin{array}{l}\text{alternate interior}\\ \text{exterior-interior}\end{array}\right\}$ **angles of parallel lines.**

Sight Exercises. Substitution Method

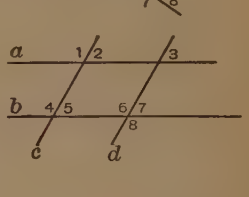
1. If $a \parallel b$, prove $\angle 2 = \angle 7$.



2. If $a \parallel b$, prove $\angle 1 + \angle 7 = 180°$.

3. If $a \parallel b$ and $\angle 2 = 2\angle 5$, prove $\angle 6 = 2\angle 1$.

In Exercises 4–7, if $a \parallel b$ and $c \parallel d$,



4. Prove $\angle 2 = \angle 7$.

5. Prove $\angle 3 = \angle 5$.

6. Prove $\angle 1 = \angle 8$.

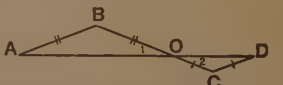
7. Prove $\angle 2 + \angle 6 = 180°$.

8. In the figure, if $PS = PR$ and $GH \parallel PS$, prove $\angle 1 = \angle R$.

9. In the same figure, if $PS = PR = RS$ and $GH \parallel PS$, prove $\angle 1 = \angle P$.

10. With the same given parts as in Ex. 9, prove $\angle 1 + \angle PGH = 180°$.

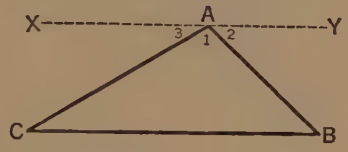
11. If $BA = BO$ and $CO = CD$, prove $\angle A = \angle D$.



12. With the same given parts as in Ex. 11, if $BO = 2\,OC$, prove $CD = \frac{1}{2}AB$.

Proposition 9 (C N)

106. Theorem. *The sum of the angles of a triangle equals a straight angle.*



Given: ABC is any triangle.

To prove: $\angle A + \angle B + \angle C = \text{st.}\angle$.

HINT: Use the Substitution Method (§ 103). What previous theorem relates to the sum of angles equal to a straight angle?

Proof: STATEMENTS	FACTS
1. Through A draw $XY \parallel CB$.	1. *Prin.* 14.
2. $\angle 1 + \angle 2 + \angle 3 = \text{st.}\angle$.	2. § 54.
3. $\angle 1 = \angle A$.	3. Why?
4. $\angle 2 = \angle B$.	4. Why?
5. $\angle 3 = \angle C$.	5. Why?
6. $\angle 1 + \angle 2 + \angle 3 = \angle A + \angle B + \angle C$.	6. Axiom of Addition.
7. $\angle A + \angle B + \angle C = \angle 1 + \angle 2 + \angle 3 = \text{st.}\angle$.	7. Sub.

NOTE: The statement of the *given* does not appear in the proof for it is assumed that the proof deals with the angles of $\triangle ABC$.

NOTE: The three principal steps of the Substitution Method as described in § 103 are Statements 2, 6, and 7.

107. Corollary 1. *The sum of two angles of a triangle is equal to a straight angle minus the third angle.*

HINT: Use § 106 and *Prin.* 6.

108. Corollary 2. *The angles adjacent to the hypotenuse of a right triangle are complementary.*

HINT: Use § 107.

Oral Exercises

In the following Exercises 1 to 7 find the values of the angles of $\triangle ABC$ if,

1. $\angle A + \angle B = 108°$, and $\angle B = \angle C$.

2. $\angle A : \angle B : \angle C = 1 : 2 : 3$.

3. $\angle A : \angle B : \angle C = 1 : 1 : 2$.

4. $\angle A = \angle B = \angle C$.

5. $\angle A = \angle B = 2\ \angle C$.

6. ABC is isosceles, and $\angle C = \angle A + \angle B$.

7. $\angle A$ and $\angle B$ are complementary, and $\angle A = 2\ \angle B$.

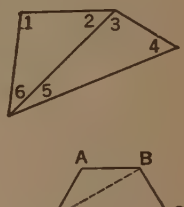
8. If $\angle X$ is increased in value how does $\angle X + \angle Y$ change?

9. In $\triangle ABC$, if $\angle A$ is decreased by 10° and $\angle B$ is increased by 20° how does $\angle C$ change?

10. How many right angles may a triangle have?

11. May each of the angles of a triangle be assigned any values? Why?

12. Are the angles of a triangle dependent or independent quantities?

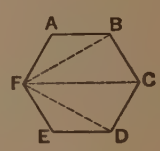


13. In the annexed figure, find the value of $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 + \angle 6$.

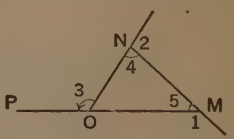
14. In the same figure, if $\angle 2 + \angle 3 + \angle 5 + \angle 6 = 200°$, what is the value of $\angle 1 + \angle 4$?

15. In the figure $ABCDEF$, find the sum of all the interior angles.

16. If the same figure is equiangular what is the value of one of its angles?

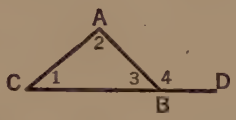


109. Exterior Angle of a Polygon. An angle formed by a side of a polygon and the prolongation of an intersecting side is called an *exterior angle* of the polygon. In the adjoining figure, $\angle 3$ is an exterior angle of $\triangle MNO$, $\angle 4$ and $\angle 5$ are called non-adjacent interior angles to $\angle 3$. $\angle 1$, $\angle 2$, and $\angle 3$ are exterior angles of $\triangle MNO$ formed by producing the sides in the same order.



Proposition 10 (c n)

110. Theorem. *An exterior angle of a triangle equals the sum of the non-adjacent interior angles.*



Given: $\angle 4$ is an exterior angle at B of $\triangle ABC$.

To prove: $\angle 4 = \angle 1 + \angle 2$.

HINT: Use the Substitution Method.

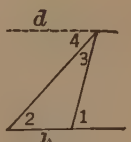
Proof:

STATEMENTS	FACTS
1. $\angle 4 + \angle 3 =$ st. $\angle$.	1. Given.
2. $\angle 4 =$ st. $\angle - \angle 3$.	2. Axiom of Subtraction.
3. st. $\angle - \angle 3 = \angle 1 + \angle 2$.	3. § 107.
4. $\angle 4 =$ st. $\angle - \angle 3 = \angle 1 + \angle 2$.	4. Sub.

NOTE: Statement 4 is a continued equation of three parts. The first and the third parts are equal because each equals the second.

Sight Exercises

1. In the adjoining figure, if $\angle 1$ is an exterior angle of the triangle, prove $\angle 1 = \angle 2 + \angle 3$ by drawing $d \parallel b$ and showing $\angle 1 = \angle 3 + \angle 4$ and $\angle 4 = \angle 2$.



2. In the adjoining figure, find the value of $\angle 1 + \angle 2 + \angle 3$.

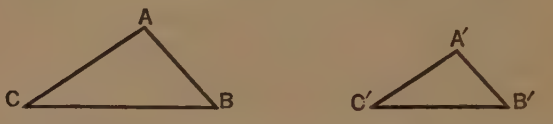
3. In the figure of § 110, if $\angle 2 = 100°$ and $\angle 4 = 140°$, prove $\angle 1 = \angle 3$.

4. Find the angles of a triangle which are in the ratio 2:2:5.

5. Name nine Facts for proving angles equal.

Proposition 11

111. Theorem. *If two angles of a triangle are equal to the respective parts of another, the third angles are equal.*



Given: $\angle A = \angle A'$, $\angle B = \angle B'$ in $\triangle ABC$ and $A'B'C'$.

To prove: $\angle C = \angle C'$.

HINT: Use the Axiom of Subtraction.

Proof: STATEMENTS	FACTS
1. $\angle A = \angle A'$.	1. Why?
2. $\angle B = B'$.	2. Why?
3. $\angle A + \angle B = \angle A' + \angle B'$.	3. Why?
4. $\angle A + \angle B + \angle C =$ st. $\angle$.	4. Why?
5. $\angle A' + \angle B' + \angle C' =$ st. $\angle$.	5. Why?
6. $\angle A + \angle B + \angle C = \angle A' + \angle B' + \angle C'$.	6. Why?
7. $\angle C = \angle C'$.	7. Why?

112. Corollary 1. *Triangles with a side and any two angles of one equal to the respective parts of another are congruent.* (*s.a.a.* = *s.a.a.*)

HINT: Use § 111 to show that *a.s.a.* = *a.s.a.*

113. Corollary 2. *Right triangles with either acute angle and a leg of one equal to the respective parts of another are congruent.* (*a.l.* = *a.l.*)

NOTE: This statement is more general than that of § 72 and should now be used in its place.

Exercises. Angles of Triangles

1. How many degrees in the angle formed by the bisectors of the acute angles of a right triangle?

2. The base angles of an isosceles triangle are bisected by lines meeting at P. How many degrees in the angle at P, if the angle at the vertex of the triangle is 40°? 60°? 80°? 45°? $x°$?

3. A, B, and C are the angles of a triangle. Find the number of degrees in each angle under the following relations:

	a	b	c	d	e	f	g
$\angle A$	$x°$	$x°$	$x°$	$x°$	$x°$	$x°$	$x°$
$\angle B$	$2x°$	$3x°$	$\frac{1}{2}x°$	$2x°$	$(x+20)°$	$(x-30)°$	$(2x-5)°$
$\angle C$	$3x°$	$5x°$	$\frac{1}{4}x°$	$\frac{1}{2}x°$	$(x+40)°$	$(x+60)°$	$(3x+35)°$

4. Find the three angles of a triangle if an exterior angle and a non-adjacent interior angle are respectively 120° and 50°; 130° and 80°; $122\frac{1}{2}°$ and $52\frac{1}{2}°$.

5. Find the three angles of a triangle if the sum and the difference of two angles are respectively 100° and 30°; 80° and 60°; $42\frac{1}{2}°$ and $37\frac{1}{2}°$; 53° and 26°.

HINT: $x + y = 100$, $x - y = 30$, $x + y + z = 180$.

6. One of the equal angles of an isosceles triangle is five-ninths of a right angle. Show that the angle at the vertex cannot be a right angle.

7. Each base angle of an isosceles triangle is one-half an exterior angle at the vertex.

8. The bisectors of two exterior angles of a triangle include an angle which equals one-half the third exterior angle.

9. Perpendiculars from a point within an angle to the sides include an angle which is the supplement of the given angle.

10. If the sides of an equilateral triangle are produced through the vertices, the sum of all the angles without the triangle equals five times the sum of the interior angles of the triangle.

11. Write an exercise of your own and solve or prove it.

(Supplementary Exercises 1–11, p. 111, may be studied now.)

Proposition 12 (c N)

114. Theorem. *Lines forming equal alternate interior angles with a transversal are parallel.*

Given: a and b with transversal t making $\angle 1 = \angle 2$.

To prove: $a \parallel b$.

Proof: STATEMENTS	FACTS
1. Assume that a and b are not parallel and that they meet at some point P.	1. § 97.
Then in the triangle formed,	
2. $\angle 1 = \angle 2 + \angle P$.	2. § 110.
3. But $\angle 1 = \angle 2$.	3. Given.

Since the assumption that a and b are not parallel leads to an absurdity, they cannot intersect and therefore $a \parallel b$.

115. Reductio Ad Absurdum Method. Another indirect form of proof (see also § 100) is given in the preceding section. The Latin phrase, *reductio ad absurdum*, means "reduction to an absurdity." The method consists of assuming the negative statement of the conclusion as true and showing that it leads to a statement which is contrary to fact. In the proof above, *we assume* ***a*** *not parallel to* ***b***. This leads us to $\angle 1 = \angle 2 + \angle P$ which is contrary to fact, since $\angle 1 = \angle 2$ by hypothesis. Therefore our assumption is false and ***a*** *must be parallel to* ***b***.

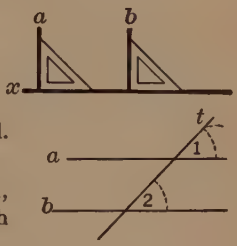
116. Corollary 1 (c n). *Lines forming equal exterior-interior angles with a transversal are parallel.*

117. Corollary 2 (c n). *Lines forming supplementary interior angles on the same side of a transversal are parallel.*

118. Corollary 3. *Two lines in the same plane parallel to the same line are parallel.* (This is Principle 16.)

HINT: Draw a transversal and use § 116.

Oral Exercises



1. The accompanying sketch shows how two lines may be drawn parallel. Why are a and b parallel?

2. The accompanying sketch shows another method for drawing two lines parallel. Why are lines a and b parallel?

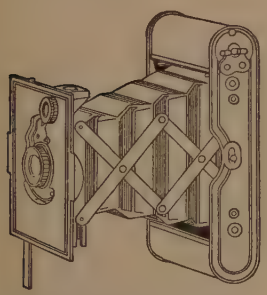
3. Explain by the use of the figure for Ex. 1, how two lines may be drawn parallel to each other.

4. Explain by the use of the figure for Ex. 2, how to draw two parallel lines.

119. Tool VII. Two lines may be proved parallel by showing that when cut by a transversal:

a) two alternate interior angles are equal, or

b) two interior angles on the same side of the transversal are supplementary.



The plane of the lens of a camera must be parallel to the plate or film when an exposure is made. Draw a diagram and show how this is possible when the camera is fully extended.

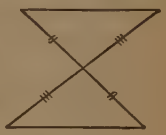
The three upright supports of the luggage carrier shown are parallel to each other. Why? Would they remain parallel if the carrier were collapsed? Of what use is the slot at the top of each upright?

Exercises. Parallel Lines

1. The bisectors of the alternate interior angles of parallel lines are parallel.

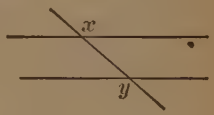
2. If two equal angles have one pair of sides parallel, the other pair of sides is parallel.

3. When two lines in the same plane are cut by a transversal, if the exterior angles on the same side of the transversal are supplementary, the lines are parallel.



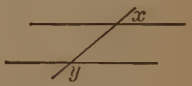
4. If two sides of a triangle are produced their own lengths through the common vertex, a line joining their ends is parallel to the third side of the triangle.

5. If two angles of a triangle are equal, the bisector of the remote exterior angle is parallel to the opposite side of the triangle.



6. If two parallel lines are cut by a transversal, the alternate exterior angles are equal.

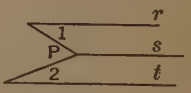
7. If two parallels are cut by a transversal, the exterior angles on the same side of the transversal are supplementary.



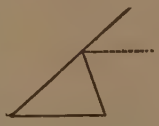
8. If a line through the vertex of an isosceles triangle makes equal angles with the sides, the line is parallel to the base of the triangle.

9. A line is drawn terminated by two parallels. Through its mid-point another line is drawn terminated by the parallels. Prove that the second line is bisected by the first.

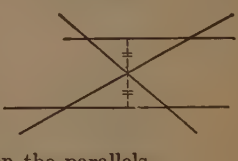
10. In the figure, $r \parallel s \parallel t$, prove $\angle 1 + 2 = \angle P$.
HINT: Prolong s through P.



11. Prove by the annexed figure: An exterior angle of a triangle equals the sum of the non-adjacent interior angles.



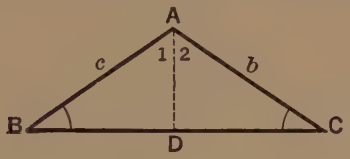
12. If through a point equidistant from two parallel lines, two lines are drawn cutting the parallels, they intercept equal line segments on the parallels.



(Supplementary Exercises 12–20, p. 112, may be studied now.)

Proposition 13 (C N)

120. Theorem. *A triangle with two angles equal is isosceles.*



Given: $\triangle ABC$ with $\angle B = \angle C$.

To prove: $c = b$.

HINT: See Hint § 79, p. 52.

Proof: STATEMENTS	FACTS
1. Draw t bisecting $\angle A$ and meeting BC at some point D.	1. *Prin.* 25.
In $\triangle ABD$ and $\triangle ACD$,	
2. $\angle B = \angle C$.	2. Given.
3. $\angle 1 = \angle 2$.	3. Cons.
4. $t = t$.	4. Iden.
5. $\triangle ABD = \triangle ACD$.	5. *s.a.a.* = *s.a.a.*
6. $c = b$.	6. *Prin.* 38.

121. Corollary. *An equiangular triangle is equilateral.*

122. Tool VIII. Two line segments may be proved equal by showing that they are sides opposite equal angles *in the same triangle*.

Oral Exercises

1. State three Facts for proving lines equal.

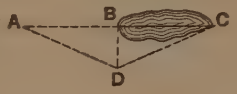
2. Complete the statement, "In the same triangle, angles opposite equal sides ———————."

3. Complete the statement, "In the same triangle, sides opposite equal angles ———————."

Exercises. Isosceles Triangles

1. If the two equal sides of an isosceles triangle are produced equal lengths through their common vertex to D and E (as in the figure), prove $DC = EB$.

2. Describe how a surveyor may find the distance BC across a lake.



3. A line perpendicular to the bisector of an angle cuts off an isosceles triangle.

4. The line which joins the vertices of two isosceles triangles on the same base is the perpendicular bisector of the common base.

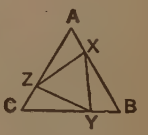
5. If the mid-points of the sides of an equilateral triangle are joined in order, four other equilateral triangles are formed.

6. If the equal sides of an isosceles triangle ABC are trisected so that $AG = GH = HB$ and $AE = ED = DC$, prove $EH = DG$.

HINT: EH and DG are corresponding sides of two triangles.

7. Two isosceles triangles are congruent if any side and angle of the one are equal to the respective parts of the other.

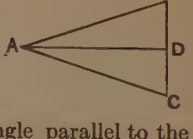
8. Every point equidistant from the ends of a line lies in the perpendicular bisector of that line.



9. If $\triangle ABC$ is equilateral, and $AX = BY = CZ$, prove $\triangle XYZ$ is equilateral.

10. If two congruent right triangles are joined as shown in the figure, show that ABC is an isosceles triangle.

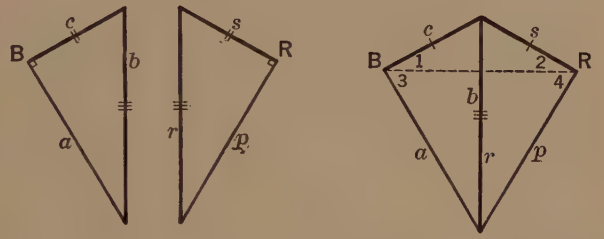
HINT: Prove $AB = AC$ and CDB is a straight line.



11. A line through two sides of an isosceles triangle parallel to the third side forms another isosceles triangle.

Proposition 14 (C N)

123. Theorem. *Right triangles with the hypotenuse and leg of one equal to the respective parts of another are congruent.* (*hy.l.* = *hy.l.*)



Given: $\triangle abc$ and prs, $\angle B = \angle R =$ rt. $\angle$, $c = s$, $b = r$.

To prove: rt. $\triangle abc \cong$ rt. $\triangle prs$.

HINT: What additional equality of parts is needed to prove the triangles congruent by *l.l.* = *l.l.*?

Proof: STATEMENTS	FACTS
1. Place rt. $\triangle abc$ so that b falls upon its equal r and $\angle B$ lies opposite $\angle R$.	1. *Prin.* 37.
2. Draw BR.	2. *Prin.* 13.
3. $c = s$.	3. Given.
4. $\angle 1 = \angle 2$.	4. Why?
5. $\angle B = \angle R$.	5. Given.
6. $\angle 3 = \angle 4$.	6. Axiom of Subtraction.
7. $a = p$.	7. Why?
8. rt. $\triangle abc \cong$ rt. $\triangle prs$.	8. *l.l.* = *l.l.*

Oral Exercises

1. Name four cases of congruence which we have proved for general triangles.

2. Name four cases of congruence which we have proved for right triangles.

3. In the preceding proof (§ 123) what other theorems may be given as the Fact for Statement 8?

SUMMARY—CONGRUENCE THEOREMS

124. The Principal Parts of a Triangle. The three sides and the three angles of a triangle are called the principal parts of the triangle.

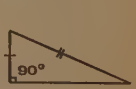
125. The Ambiguous Case of Congruence. There is a possible combination of equalities of principal parts which we have not considered in proving two triangles congruent, namely, *s.s.a.* = *s.s.a.* This case may or may not determine the congruence of triangles and is therefore called the *ambiguous case.* The following groups of figures will illustrate the uncertainty of this case. In these groups of figures equal parts are marked alike.

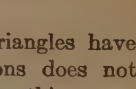
GROUP A
These triangles *are* congruent.

GROUP B
These triangles *are not* congruent.

GROUP C
These triangles *are* congruent.

GROUP D
These triangles *are* congruent.



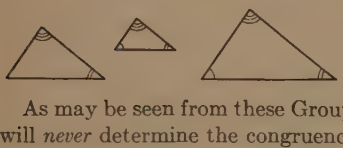


In each of the Groups A, B, C, and D the two triangles have *s.s.a.* = *s.s.a.* As may be seen, this set of conditions does not *always* determine the congruence of the triangles and for this reason may not be used to prove congruence.

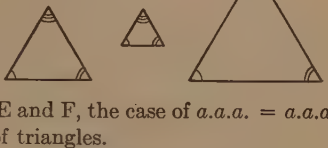
126. The Impossible Case. Another combination of equalities of principal parts of two triangles is *a.a.a.* = *a.a.a.*, which is often called the *impossible case.*

The following groups of figures illustrate that triangles with three angles of one equal to three angles of another are not necessarily congruent:

GROUP E



GROUP F



As may be seen from these Groups E and F, the case of *a.a.a.* = *a.a.a.* will *never* determine the congruence of triangles.

127. Four Cases of Congruence for General Triangles. There are only four other possible combinations of equalities of principal parts of two triangles. They are:

(1) *s.a.s.* = *s.a.s.* (§ 65);
(2) *a.s.a.* = *a.s.a.* (§ 71);
(3) *s.s.s.* = *s.s.s.* (§ 84);
(4) *s.a.a.* = *s.a.a.* (§ 112).

Each of these cases has been previously proved and has been used many times in our past work.

128. Four Cases of Congruence for Right Triangles. Of the various kinds of special triangles with which we deal in geometry, the right triangle is the most important and the most used. For this reason, it is advisable for us to keep in mind the ways by which we may show right triangles congruent.

We have proved the following four cases:

(1) *l.l.* = *l.l.* (§ 69);
(2) *hy.a.* = *hy.a.* (§ 90);
(3) *a.l.* = *a.l.* (§ 113);
(4) *hy.l.* = *hy.l.* (§ 123).

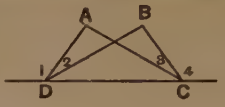
In each of these cases, the right angles, of course, are known to be equal. It should be further observed *in each case of congruence* which we use that there are three pairs of equal parts known and that at least *one* such *pair must be sides.*

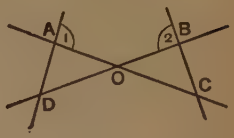
129. *Triangles congruent to the same triangle are congruent to each other.* This is true because their parts are respectively equal.

130. The Use of Congruence. The principal use to which congruence is put is to prove the equality of a pair of corresponding angles or a pair of corresponding sides of two triangles. This use was emphasized in Tool III (§ 78).

Review Exercises. Congruent Triangles

1. In the figure, if $\angle 1 = \angle 4$ and $\angle 2 = \angle 3$, prove $\triangle ADC \cong \triangle BCD$.

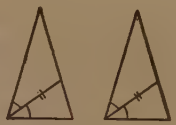




2. If $DO = OC$ and $\angle 1 = \angle 2$, prove $\triangle AOD \cong \triangle COB$.

3. In the figure, if $\angle A = \angle B$, $CD = CF$, and $AE = EB$, prove $\triangle DEA \cong \triangle FEB$.

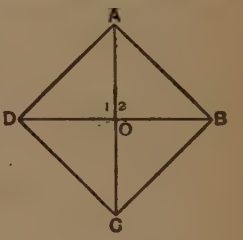
4. Two isosceles triangles with a base angle and a bisector of a base angle of one equal to the respective parts of the other are congruent.



5. In the figure, if $AC = AB$ and $OC = OB$, prove $\triangle ODC \cong \triangle BDO$.

6. In the same figure, if AD is the prependicular bisector of BC, prove $\triangle AOC \cong \triangle ABO$.

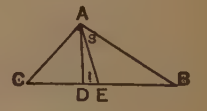
7. Show that, two right triangles are congruent if any two sides of the one equal the respective parts of the other.



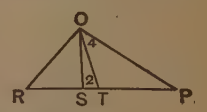
8. In the figure, if $\angle 1 = \angle 2$, $AO = OC$, and $DO = OB$, prove $AB = BC = CD = DA$.

9. If from a point in a perpendicular to a line, two lines are drawn which cut off equal segments on the line from the foot of the perpendicular, these lines are equal.

10. In the figures, if $\angle 1 = \angle 2 =$ rt. $\angle$, $CE = EB$, $RT = TP$, $AD = OS$, $AE = OT$, and $AB = OP$, prove $\triangle ABC \cong \triangle OPR$.



11. In the same figures, if in place of $AB = OP$ we take $\angle B = \angle P$, prove $\triangle ABC \cong \triangle OPR$.

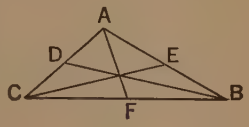


(Supplementary Exercises 21–34, p. 112, may be studied now.)

131. The Secondary Parts of a Triangle. The bisectors of the angles, the perpendicular bisectors of the sides, the

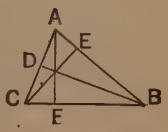
medians, and the altitudes are called the *secondary parts* of a triangle.

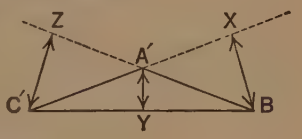
132. Median. The line from a vertex of a triangle to the mid-point of the opposite side is called a *median* of the triangle.



In the annexed figure, AF, BD, and CE are medians.

133. Altitude. The line from a vertex of a triangle perpendicular to the opposite side or its prolongation is called an *altitude* of the triangle. In the figures, AF, BD, CE, $A'Y$, $B'X$, and $C'Z$ are altitudes of $\triangle ABC$ and $\triangle A'B'C'$.





Drawing Exercises

1. Sketch an isosceles triangle and its altitudes. How many altitudes has a triangle? Do they appear to meet in the same point?

2. Sketch a right triangle and its altitudes.

3. In the adjoining figure, sketch the median to BC, the altitude to BC, the bisector of $\angle A$, and the perpendicular bisector of BC.

4. Sketch an equilateral triangle and all of its secondary parts. Letter and name all these parts.

Exercises. Perpendiculars, Medians, and Angle Bisectors

1. The altitude of an isosceles triangle bisects the base. Prove.

2. In an isosceles triangle the perpendiculars drawn from the extremities of the base to the opposite sides are equal.

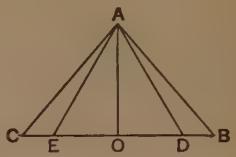
3. In an isosceles triangle the lines perpendicular to the base from the mid-points of the two equal sides are equal.

4. The corresponding altitudes of two congruent triangles are equal.

5. If two altitudes of a triangle are equal, the triangle is isosceles.

6. The altitude upon the hypotenuse of a right triangle forms two right triangles whose angles are respectively equal.

7. In the figure, AO is the altitude of the isosceles triangle ABC and $CE = BD$; prove that AO forms equal angles with AE and AD.



8. The median of an isosceles triangle is perpendicular to the base.

9. The median to the base of an isosceles right triangle is equal to half the base.

10. If the median to the base of a triangle is equal to half the base, is the triangle an isosceles right triangle?

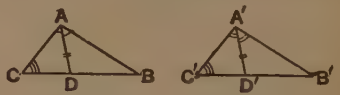
11. The medians of an equilateral triangle are equal.

12. The corresponding medians of two congruent triangles are equal.

13. If the bisector of an angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.

14. The corresponding angle bisectors of two congruent triangles are equal.

15. In the figure, AD and $A'D'$ are angle bisectors and are equal, $\angle A = \angle A'$ and $\angle C = \angle C'$. Prove $\triangle ADB \cong \triangle A'D'B'$.



Oral Exercises

1. Draw an acute angle and call its sides OA and OB.

2. Draw a line parallel to OA and call it MN. Draw a line parallel to OB intersecting MN at P and call this line RS.

3. How many angles of the second figure have their sides respectively parallel to those of the first figure?

4. If an angle is observed from its vertex, its sides are often called "the right side" and "the left side." Using these terms, tell which angles of the two figures appear to be equal; supplementary.

5. Are OB and MN parallel? If produced, would they intersect? (§ 97.)

Proposition 15 (c n)

134. Theorem. *Angles whose sides are parallel each to each are either equal or supplementary.*

Given: Angles 2 and 4 have sides parallel to those of $\angle 1$.

To prove: $\angle 1 = \angle 2$ and $\angle 4 + \angle 1 = \text{st.}\angle$.

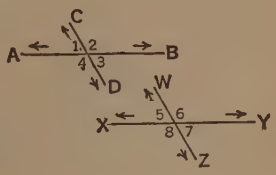
HINT: Use the Substitution Method. What equality of angles do parallel lines suggest?

Proof: STATEMENTS	FACTS
1. Produce the sides of the angles until a pair of sides intersect, forming $\angle 3$.	1. *Prin.* 11.
2. $a \parallel c$ and $b \parallel d$.	2. Given.
3. $\angle 1 = \angle 3$.	3. Why?
4. $\angle 3 = \angle 2$.	4. Why?
5. $\angle 1 = \angle 3 = \angle 2$.	5. Why?
6. $\angle 4 + \angle 2 = \text{st.}\angle$.	6. Why?
7. $\angle 4 + \angle 1 = \text{st.}\angle$.	7. Why?

Oral Exercises

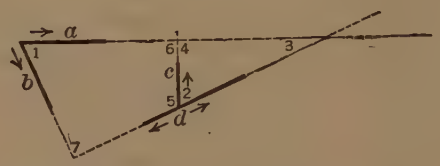
1. In the figure of § 134, describe the relative positions of the sides of the equal angles; of the supplementary angles.

2. In the annexed figure, if $AB \parallel XY$ and $CD \parallel WZ$, name at least eight pairs of equal angles; at least ten pairs of supplementary angles.



Proposition 16 (c n)

135. Theorem. *Angles whose sides are perpendicular each to each are either equal or supplementary.*



Given: Angles 2 and 5 have their sides perpendicular to those of $\angle 1$.

To prove: $\angle 1 = \angle 2$ and $\angle 1 + \angle 5 = \text{st.}\angle$.

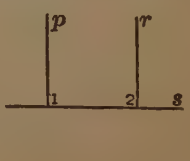
HINT: What angles of a right triangle are complementary?

Proof: STATEMENTS	FACTS
1. Produce the sides of the angles until they intersect.	1. *Prin.* 11.
2. $\angle 4 = \angle 7 = \angle$ rt.$\angle$.	2. $c \perp a$ and $d \perp b$. (Given.)
3. $\angle 1$ is the complement of $\angle 3$.	3. Why?
4. $\angle 2$ is the complement of $\angle 3$.	4. Why?
5. $\angle 1 = \angle 2$.	5. Why?
6. $\angle 2 + \angle 5 = \text{st.}\angle$.	6. Why?
7. $\angle 1 + \angle 5 = \text{st.}\angle$.	7. Why?

Oral Exercises

1. In drawing figures to represent a pair of congruent triangles how may § 134 or § 135 be used to aid in drawing the corresponding sides?

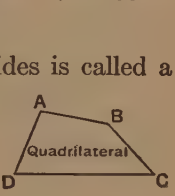
2. In the annexed figure, if p and r are perpendicular to s, why are $\angle 1$ and $\angle 2$ equal and supplementary?



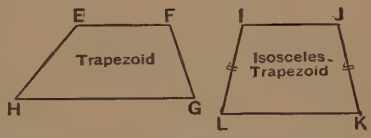
QUADRILATERALS

A polygon has been defined as a closed broken line. The numerical measure of a polygon is called its *perimeter*. It is equal to the sum of the lengths of the sides. (See §§ 58 and 60.)

136. Quadrilateral. A polygon of four sides is called a *quadrilateral*. A quadrilateral with no two sides parallel is the *general quadrilateral*. *ABCD* is a quadrilateral.

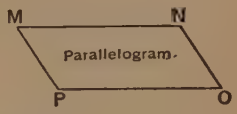


137. Trapezoid. A quadrilateral with two sides parallel is called a *trapezoid;* if the non-parallel sides are equal, it is called an *isosceles trapezoid*. *EFGH* is a trapezoid. *IJKL* is an isosceles trapezoid.

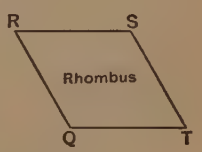


138. Parallelogram. A quadrilateral with its opposite sides parallel is called a *parallelogram*. (□)

A parallelogram with any angle acute or obtuse is the *general parallelogram*. *MNOP* is a parallelogram.

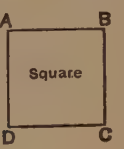


139. Rhombus. A parallelogram with a pair of intersecting sides equal is called a *rhombus*. *RSTQ* is a rhombus.



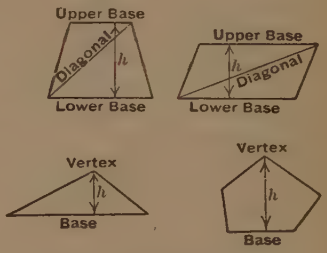
140. Rectangle. A parallelogram with a right angle is called a *rectangle*. *UVWX* is a rectangle.

141. Square. A rectangle with a pair of intersecting sides equal is called a *square*. *ABCD* is a square.



142. Bases and Vertices of a Polygon. A polygon has as many angles as it has sides. Any side of a polygon may be called its *base*. If a polygon has two sides parallel both may be called *bases;* one, the *upper base*, the other, the *lower base*. If a polygon has a vertex opposite a side called the *base*, such point may be called the *vertex of the polygon*.

143. Altitude of a Polygon. The line which is perpendicular to the bases, or, which is perpendicular from a vertex to the respective base, is called an *altitude of the polygon*. The arrowed lines marked h in the annexed figures are altitudes.



144. Diagonal of a Polygon. The line which joins any two non-consecutive vertices of a polygon is called a *diagonal of a polygon*. (*Vertices* in this sense means the "vertices of the angles of a polygon.")

Oral Exercises

1. What are the common properties of a trapezoid and a parallelogram?

2. Compare a square and a rhombus.

3. Is a square a rectangle? Is a rectangle a square?

4. How many bases may a polygon have? How many altitudes?

5. What is a diagonal? How many diagonals has a parallelogram?

6. Which side of a rectangle may be taken as its altitude? Why?

7. A diagonal of a trapezoid divides it into what kind of figures?

8. A diagonal of a rectangle divides it into what kind of figures?

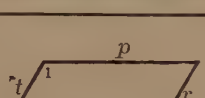
9. What special figure is a quadrilateral? a parallelogram? a rectangle? a square?

10. What two special properties does an isosceles trapezoid possess?

11. Name the equilateral parallelograms.

Proposition 17 (n c)

145. Theorem. *The opposite angles of a parallelogram are equal and any two consecutive angles are supplementary.*



Given: $\square\, prst$, $p \parallel s$, $r \parallel t$.

To prove: $\angle 1 = \angle 2$, $\angle 3 + \angle 2 = \text{st.}\angle$.

HINT: Under what conditions do parallel lines make equal angles?

Proof: STATEMENTS	FACTS
1. $p \parallel s$ and $r \parallel t$.	1. Why?
2. $\angle 1 = \angle 2$.	2. Why?
3. $\angle 3 + \angle 2 = \text{st.}\angle$.	3. Why?

Proposition 18 (N)

146. Theorem. *Either diagonal of a parallelogram divides it into two congruent triangles.*

(The pupil may complete the demonstration.)

147. Corollary 1 (c n). *The opposite sides of a parallelogram are equal.*

148. Corollary 2. *Segments of parallels cut off between parallels are equal.*

Proposition 19 (c n)

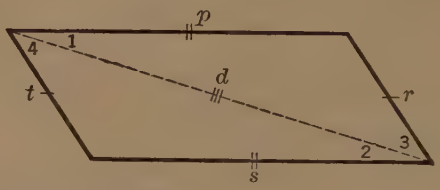
149. Theorem. *The diagonals of a parallelogram bisect each other.*

(The pupil may complete the demonstration.)

150. Tool IX. Two $\left\{\begin{matrix}\text{angles}\\\text{lines}\end{matrix}\right\}$ may be proved equal by showing that they are opposite $\left\{\begin{matrix}\text{angles}\\\text{sides}\end{matrix}\right\}$ of a parallelogram.

Proposition 20 (C N)

151. Theorem. *A quadrilateral with its opposite sides equal is a parallelogram.*



Given: Quadrilateral *prst*, $p = s$, $r = t$.

To prove: *prst* is a $\square$.

HINT: What is a parallelogram? How are lines proved parallel?

Proof: STATEMENTS	FACTS
1. Draw diagonal d. In $\triangle prd$ and std,	1. By what Principle?
2. ? = ?	2. Why?
3. ? = ?	3. Why?
4. ? = ?	4. Why?
5. ? $\cong$?	5. *s.s.s.* = *s.s.s.*
6. $\angle 1 = \angle 2$.	6. Why?
7. $p \parallel s$.	7. Why?
8. $\angle 3 = \angle 4$.	8. Why?
9. $r \parallel t$.	9. Why?
10. *prst* is a $\square$.	10. (*Def.*) § 138.

Proposition 21

152. Theorem. *A quadrilateral is a parallelogram if*

a) two sides are both equal and parallel; **(C N)**

b) the diagonals bisect each other; **(n)**

c) the opposite angles are equal. **(n)**

(The pupil may complete the demonstration.)

153. Tool X. A quadrilateral may be proved to be a parallelogram by showing that

***a*) *two pairs* of opposite sides are *parallel*, or**

***b*) *two pairs* of opposite sides are *equal*, or**

***c*) *one pair* of opposite sides are *equal and parallel*.**

Exercises. Parallelograms

1. If two opposite sides of a parallelogram are produced equal lengths in the same direction and their extremities joined, another parallelogram is formed.

2. If the bisectors of two opposite angles of a parallelogram are terminated by the sides, the bisectors are equal and parallel.

3. The perpendiculars drawn from two opposite vertices of a parallelogram to their opposite sides are equal and parallel.

4. The diagonals of a rectangle are equal.

5. The diagonals of a rhombus are perpendicular.

6. If the diagonals of a parallelogram are perpendicular, the figure is equilateral.

7. The bisectors of the four angles of a parallelogram enclose a rectangle.

8. The diagonals of a rhombus divide the figure into four congruent triangles.

9. The perpendiculars drawn from two opposite vertices of a parallelogram to a diagonal are equal and parallel.

10. If the mid-points of the halves of each diagonal of a parallelogram are connected in order, another parallelogram is formed.

11. If a diagonal bisects an angle of a parallelogram, the figure is equilateral.

12. If one angle of a parallelogram is a right angle, the other three angles are right angles.

13. Draw a parallelogram with two diagonals and indicate

a) four pairs of equal lines,

b) eight pairs of equal angles,

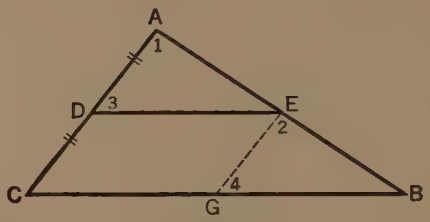
c) four pairs of congruent triangles,

d) eight pairs of supplementary angles.

(Supplementary Exercises 35–44, p. 113, may be studied now.)

Proposition 22

154. Theorem. *A line which is parallel to one side of a triangle and bisects a second side bisects the third side.*



Given: $DE \parallel CB$, $AD = DC$ in $\triangle ABC$.

To prove: $AE = EB$.

HINT: How are two line segments usually proved equal?

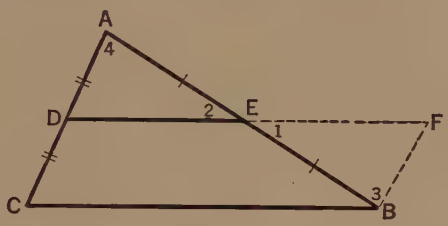
Proof: STATEMENTS	FACTS
1. Through E draw $EG \parallel AC$.	1. *Prin.* 14.
2. $DE \parallel CB$.	2. Why?
3. $EG = CD$.	3. § 148.
4. $AD = CD$.	4. Why?
5. $AD = CD = EG$.	5. Why?
6. $\angle 1 = \angle 2$.	6. § 102.
7. $\angle 3 = \angle 4$.	7. Why?
8. $\triangle ADE \cong \triangle EGB$.	8. *a.s.a.* = *a.s.a.*
9. $AE = EB$.	9. Why?

Oral Exercises (based on § 154)

1. Why is GE drawn $\parallel AC$?
2. Find eight pairs of equal angles in the figure.
3. Can you prove $\triangle ADE \cong \triangle EGB$ in more than one way?
4. Why does $CG = GB$?
5. Why is $DEGC$ a parallelogram?

Proposition 23

155. Theorem. *A line which bisects two sides of a triangle is parallel to the third side and equal to half of it.*



Given: $AD = DC$, $AE = EB$ in $\triangle ABC$.

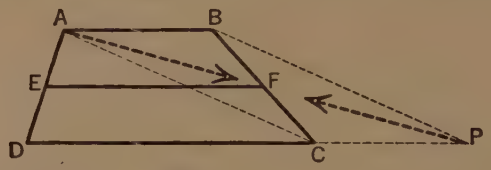
To prove: $DE \parallel CB$ and $DE = \frac{1}{2} CB$.

HINT: The opposite sides of a parallelogram are parallel. A line may be proved to be half of another by showing that double the shorter equals the longer.

Proof: STATEMENTS	FACTS
1. Produce DE making $EF = DE = \frac{1}{2} DF$.	1. *Prin.* 11.
2. Draw FB.	2. *Prin.* 13.
3. $\angle 1 = \angle 2$.	3. *Prin.* 27.
4. $AE = EB$.	4. Given.
5. $\triangle FEB \cong \triangle DEA$.	5. *s.a.s.* = *s.a.s.*
6. $FB = AD$.	6. Why?
7. $DC = AD$.	7. Why?
8. $FB = AD = DC$.	8. Why?
9. $\angle 3 = \angle 4$.	9. Why?
10. $FB \parallel AC$.	10. Why?
11. $DFBC$ is a $\square$.	11. § 152 (a).
12. DE (or DF) $\parallel CB$.	12. Def. of parallelogram.
13. $DF = CB$.	13. Why?
14. $DE = \frac{1}{2} CB$.	14. Sub.

Proposition 24

156. Theorem. *The line which joins the mid-points of the non-parallel sides of a trapezoid is parallel to the bases and equal to half their sum.*



Given: $ABCD$ is a trapezoid, $AE = ED$, and $BF = FC$.

To prove: $EF \parallel AB \parallel DC$ and $EF = \frac{1}{2}(AB + DC)$.

HINT: Use § 155 and the Substitution Method.

Proof:

STATEMENTS	FACTS
1. Produce DC to make $CP = AB$.	1. *Prin.* 11.
2. Draw BP, AP, and AC.	2. *Prin.* 13.
3. $ABPC$ is a $\square$.	3. Why?
4. $AF = FP$.	4. § 149.
In $\triangle APD$,	
5. $EF \parallel DP$ (or DC).	5. § 155.
6. $DC \parallel AB$.	6. Def. of trapezoid.
7. $EF \parallel DC \parallel AB$.	7. Why?
8. $EF = \frac{1}{2} DP$.	8. Why?
9. $DP = AB + DC$.	9. Cons.
10. $EF = \frac{1}{2} DP = \frac{1}{2}(AB + DC)$.	10. Sub.

157. Median of a Trapezoid. The line which joins the mid-points of the non-parallel sides of a trapezoid is called the *median* of the trapezoid. Such a line as EF in the figure of § 156 is the median of $ABCD$.

Exercises. Trapezoids

1. A median of a trapezoid divides the figure into two other trapezoids.

2. The diagonals of an isosceles trapezoid form a pair of congruent triangles.

3. Prove that two pairs of angles of a trapezoid are supplementary.

4. The lines which join in order the mid-points of the sides of a trapezoid enclose a parallelogram.

HINT: Draw the diagonals and apply §155.

5. The diagonals of an isosceles trapezoid are equal.

6. The base angles of an isosceles trapezoid are equal.

7. The line joining the mid-points of the non-parallel sides of a trapezoid bisects the diagonals.

8. The line joining the mid-points of the non-parallel sides of a trapezoid bisects an altitude of the trapezoid.

9. If the equal sides of an isosceles trapezoid are produced to meet, two isosceles triangles are formed.

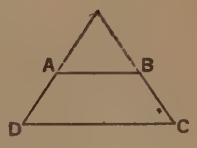
10. If the angles adjacent to one base of a trapezoid are equal those adjacent to the other base are equal.

11. The line joining the mid-points of the bases of an isosceles trapezoid is perpendicular to the bases.

12. If the base angles of a trapezoid are equal, the trapezoid is isosceles.

13. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

14. In the adjoining figures, $AD = BC$ and $\angle D = \angle C$. Prove $AB \parallel DC$ by each of the figures shown.

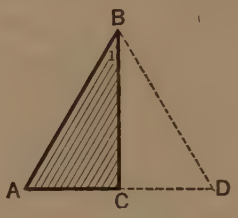


15. In the adjoining figure, $ABCD$ is a trapezoid and $AB = DF = EC$. Prove $\triangle ADE \cong \triangle BFC$.

16. Two isosceles trapezoids are congruent if their sides are respectively equal.

Proposition 25

158. Theorem. *The hypotenuse of a 30°–60°–90° triangle is twice the side opposite the 30° angle.*



Given: $\angle A = 60°$, $\angle 1 = 30°$, and $\angle C = 90°$ in $\triangle BCA$.

To prove: $AB = 2\,AC$.

HINT: Show that double the shorter leg equals the hypotenuse.

Proof: STATEMENTS	FACTS
1. Produce AC to make $AD = 2\,AC$.	1. *Prin.* 11.
2. Draw DB.	2. *Prin.* 13.
In rt. $\triangle BCA$ and BCD,	
3. $BC = BC$.	3. Identical.
4. $AC = CD$.	4. Cons.
5. $? \cong ?$	5. *l.l.* = *l.l.*
6. $\angle A = \angle D$.	6. Why ?

(The pupil may complete the demonstration.)

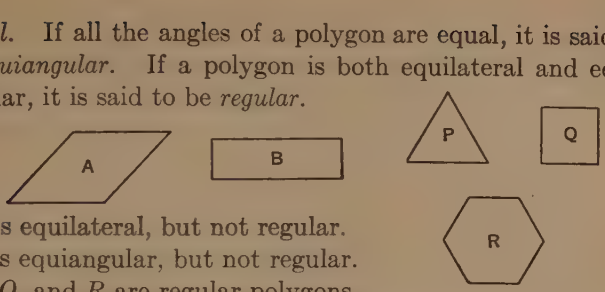
159. Theorem. *In a right triangle whose hypotenuse is twice one of its legs, the angle opposite that leg is 30° and the triangle is a 30°–60°–90° triangle.*

HINT: Use Statements 1 to 6 in § 158 and prove $\angle A = \angle D = \angle B$.

POLYGONS

160. Polygons—Equilateral, Equiangular, and Regular. If all the sides of a polygon are equal, it is said to be *equi-*

lateral. If all the angles of a polygon are equal, it is said to be *equiangular.* If a polygon is both equilateral and equiangular, it is said to be *regular.*



A is equilateral, but not regular.
B is equiangular, but not regular.
P, *Q*, and *R* are regular polygons.

161. A polygon is named according to the number of its sides, as follows:

No. of Sides	Name
Three	Trigon or Triangle
Four	Quadrigon or Quadrilateral
Five	Pentagon
Six	Hexagon
Seven	Heptagon
Eight	Octagon
Nine	Nonagon
Ten	Decagon
Eleven	Undecagon
Twelve	Dodecagon
Fifteen	Pentadecagon
N	*n*-gon (the general polygon)

162. Polygons—Mutually Equiangular; Mutually Equilateral. If the angles of a polygon are equal respectively to the angles of another polygon, taken in the same order, the polygons are said to be *mutually equiangular;* if the sides of one are equal respectively to the sides of the other, taken in the same order, the polygons are said to be *mutually equilateral.*

163. Congruent Polygons. Two polygons which are mutually equiangular and mutually equilateral can be made to coincide and are said to be congruent.

A and *B* are mutually equiangular, but are not congruent.
C and *D* are mutually equilateral, but are not congruent.
E and *F* are congruent pentagons.

Oral Exercises

1. Two triangles which are equiangular are also mutually equiangular. Explain.

2. May two scalene or two right triangles be mutually equiangular?

3. If two triangles are congruent are they mutually equiangular?

4. If two triangles are congruent are they mutually equilateral?

5. What kind of triangles is regular?

6. Criticize: A square is a mutually equiangular rectangle.

7. In § 162 why are the words, "taken in the same order" included in the definition? Are they necessary?

8. Define a regular octagon.

9. Are mutually equilateral triangles congruent?

10. Are mutually equilateral quadrilaterals congruent?

11. Are squares congruent? Are they regular?

12. What is a triangle called if its angles are in the ratio 1:2:3?

13. What is the sum of the angles of a square?

14. What is the sum of the angles of any quadrilateral?

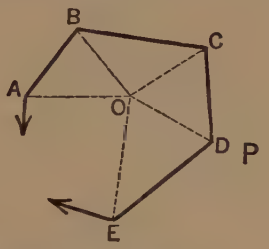
15. If a polygon has n sides, how many vertices does it have?

16. By drawing diagonals from any vertex, find the sum of the interior angles of a pentagon; a hexagon; a decagon.

17. What is the relation between the number of sides of each polygon of Ex. 16 and the sum of its angles expressed as straight angles?

Proposition 26 (c n)

164. Theorem. *The sum of the interior angles of a polygon of* n *sides equals* $(n - 2)\ 180°$.



Given: P is a polygon of n sides.

To prove: $\angle A + \angle B + \angle C + \angle D + \text{etc.} = (n-2)\ 180°$.

HINT: What previous theorem on the sum of angles has been proved?

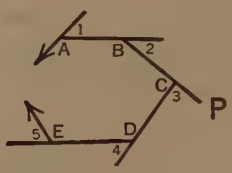
Proof: STATEMENTS	FACTS
1. Polygon P has n sides.	1. Given.
2. From any point within draw lines to the vertices forming n triangles.	2. *Prin.* 13.
3. The sum of the angles of a triangle $=$ st. $\angle$.	3. § 106.
4. The sum of the angles of n triangles $= n$ st. $\measuredangle$.	4. *Prin.* 7.
5. The sum of the angles about point O $= 2$ st. $\measuredangle$.	5. § 36.
6. $\angle A + \angle B + \angle C + \angle D +$ etc. $= n$ st. $\measuredangle - 2$ st. $\measuredangle = (n - 2)\ 180°$.	6. *Prin.* 6.

NOTE: The figure of Proposition 26 does not show the polygon as a closed broken line. It is left incomplete in order to represent a polygon of no specified number of sides, that is, any polygon or an n-gon.

165. Tool XI. The sum of the interior angles of an n-gon $= (n - 2)\ 180°$.

Proposition 27

166. Theorem. *The sum of the exterior angles of a polygon of* ***n*** *sides, if the sides are produced in the same order, equals* ***360°.***



Given: P is a polygon of n sides.

To prove: $\angle 1 + \angle 2 + \angle 3 + \angle 4 +$ etc. $= 360°$.

HINT: How is each exterior angle related to its adjacent interior angle?

Proof: STATEMENTS	FACTS
1. Polygon P has n sides.	1. Given.
2. Polygon P has n vertices.	2. § 142.
3. At any vertex such as A, $\angle A + \angle 1 = 180°$.	3. *Prin.* 29.
4. At n vertices $(\angle A + \angle B + \angle C + \angle D +$ etc.$) + (\angle 1 + \angle 2 + \angle 3 + \angle 4 +$ etc.$) = n \cdot 180°$.	4. Axiom of Addition.
5. $(\angle A + \angle B + \angle C + \angle D +$ etc.$) = (n - 2)\,180°$.	5. § 164.
6. $(\angle 1 + \angle 2 + \angle 3 + \angle 4 +$ etc.$) = n \cdot 180° - (n - 2)\,180°$ $= (n - n + 2)\,180° = 2 \times 180° = 360°$.	6. Axiom of Subtraction.

167. Corollary. *Each interior angle of a regular polygon of* ***n*** *sides equals* $\frac{(n-2)}{n}$ ***180°;*** *and each exterior angle equals* $\frac{360°}{n}$.

Exercises. Angles of Polygons

1. If the four angles of a quadrilateral are equal, each is equal to a right angle.

2. If two angles of a quadrilateral are right angles, the other two angles are supplementary.

3. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

4. Find the total number of degrees in the interior angles of a quadrilateral; of a pentagon; of a hexagon; of an octagon; of a decagon; of a polygon of 50 sides; of a polygon of 102 sides.

5. If the polygons of Ex. 4 are equiangular, find the number of degrees in an exterior angle of each polygon.

6. How many sides has a polygon if the sum of the interior angles equals 720°? 900°? 1260°? 1980°? 3600°?

7. How many sides has a polygon if the sum of the interior angles equals 8 rt. ∠s? 28 rt. ∠s? 4 st. ∠s? 10 st. ∠s? n rt. ∠s? n st. ∠s?

8. How many sides has an equiangular polygon if each of its exterior angles equals 72°? 36°? 20°? 18°? 9°? 6°? $13\frac{1}{3}$°? $10\frac{2}{7}$°? $4\frac{4}{5}$°?

9. How many sides has an equiangular polygon if each of its interior angles equals 108°? 135°? 144°? 150°? 156°? 160°? $157\frac{1}{2}$°? 165°? 171°? $171\frac{3}{7}$°? $173\frac{1}{13}$°?

10. If the sides of a regular pentagon are produced to form a star-shaped figure or pentagram, find the sum of the angles at the points.

11. Find the sum of the angles at the points of a regular hexagram; of a regular octagram.

12. Four angles of a pentagon are 95°, 90°, 100°, and 110°. Find the fifth angle.

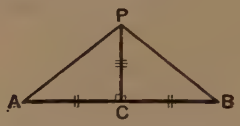
13. If the sides of the polygon of Exercise 12 are produced to form an irregular star polygon, find the sum of the angles at the five points.

14. Prove that an exterior angle of a regular polygon cannot equal 7°.

(Supplementary Exercises 45–55, p. 114, may be studied now.)

Proposition 28

168. Theorem. *Any point in the perpendicular bisector of a line is equidistant from the extremities of the line.*



Given: $PC \perp AB$, $AC = BC$, and P any point in PC.

To prove: $PA = PB$.

HINT: The distance between two points is the length of the line joining them. How may lines be proved equal?

(The pupil may complete the demonstration.)

169. Corollary. *Two lines drawn from any point in a perpendicular cutting off equal distances from the foot of the perpendicular are equal.*

Proposition 29

170. Theorem. *Any point equidistant from the extremities of a line lies in its perpendicular bisector.*

Given: AB and R so that $RA = RB$.

To prove: R is in the perpendicular bisector of AB.

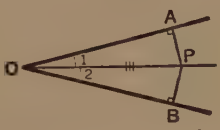
HINT: Drop a perpendicular from R to AB and show that it bisects AB. What line segments are to be proved equal?

(The pupil may complete the demonstration.)

171. Corollary. *Two points each equidistant from the ends of a line determine the perpendicular bisector of the line.*

Proposition 30

172. Theorem. *Any point on the bisector of an angle is equidistant from the sides of the angle.*



Given: $\angle 1 = \angle 2$, $PA \perp OA$, $PB \perp OB$, and P any point in OP.

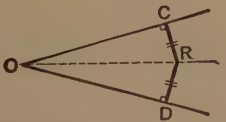
To prove: $PA = PB$.

HINT: The distance from a point to a line is the length of the perpendicular from the point to the line. How are lines usually proved equal?

(The pupil may complete the demonstration.)

Proposition 31

173. Theorem. *Any point equidistant from the sides of an angle lies on the bisector of the angle.*



Given: $\angle O$ and R so that $RC = RD$; $RC \perp OC$, $RD \perp OD$.

To prove: R is in the bisector of $\angle O$.

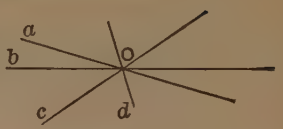
HINT: Join O and R and show that OR bisects $\angle O$.

(The pupil may complete the demonstration.)

Question. What two points determine the bisector of an angle?

CONCURRENT LINES OF A TRIANGLE

174. Concurrent Lines. Three or more lines which pass through the same point are called *concurrent* lines.



For example; a, b, c, and d are concurrent lines.

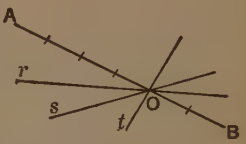
175. Three Concurrence Methods. In order to prove that three or more lines are concurrent, one of three methods is usually employed. They are as follows:

A) Two of the lines are shown to intersect and the point of intersection is definitely marked or located on one of these lines and then each of the other lines is shown to pass through that point.

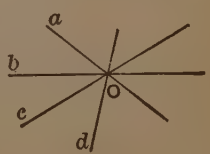
B) Sometimes it may not be possible or convenient to use Method *A* because of the difficulty in definitely locating the point of intersection on one of the lines. Under Method *B*, two of the lines are shown to intersect and then the point of intersection is shown to lie upon each of the other lines.

C) The concurrence of lines may also be established by showing (by the Corollary Method) that the given conditions embody those of a theorem which has been proved by Method *A* or *B*.

Example—Method *A*. The lines AB, r, s, and t are concurrent if it can be shown that r intersects AB so that point O is defined by $AO = 2\,OB$ and that s and t also cut AB so that $AO = 2OB$. That is, r, s, and t cut AB at O.



Example—Method *B*. The lines a, b, c, and d are concurrent if it can be shown that a and b intersect at some point such as O, and that O lies on c and d.



Example—Method *C*. (See § 178.)

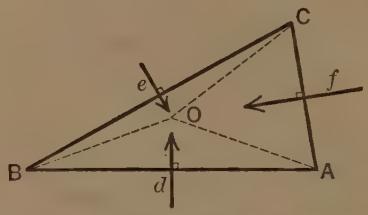
A GREENLAND GIRL

The costume of Hansine Alrick, a Greenland girl, is distinguished for its bright bead collar and fine boots. The collar is an elaborate homemade affair weighing between two and three pounds. Her red-and-white boots are made of sealskin and decorated with geometric patterns of colored skin. The most colorful part of the costume is the thigh panel on the bright sealskin trousers. Inside a border of shiny red there is a strip of white from a reindeer breast, and the inner section is a polychrome mosaic composed of hundreds of tiny bits of leather sewn to the background.

Question. If two lines are perpendicular to the same line, do they meet? Why? If two lines are perpendicular to intersecting lines (as d and e in § 176), do they meet? (*Prin.* 19.)

Proposition 32 (c n)

176. Theorem. *The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices.*



Given: d, e, and f are perpendicular bisectors of the sides of $\triangle ABC$.

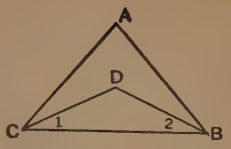
To prove: d, e, and f are concurrent (at O) and $OA = OB = OC$.

HINT: Use Concurrence Method *B*. What previous theorem relates to a perpendicular bisector?

Proof: STATEMENTS	FACTS
1. d and e intersect (at O).	1. *Prin.* 19.
2. Since O is on d, $OA = OB$.	2. § 168.
3. Since O is on e, $OB = OC$.	3. Why?
4. $OA = OB = OC$.	4. Sub.
5. Since $OA = OC$, O is on f.	5. § 170.
6. d, e, and f are concurrent.	6. § 174.

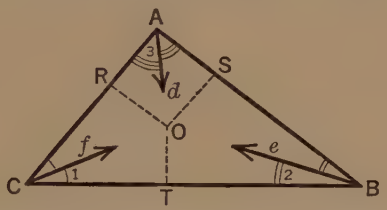
NOTE: Point O is the center of a circle which passes through A, B, and C. Such a point is often called the *circumcenter* of the triangle, because it is the center of the circumscribed circle.

Question. In the annexed figure, why cannot $\angle 1 + \angle 2$ equal 180°? If $\angle 1 + \angle 2 \neq 180°$, why do CD and BD intersect? (*Prin.* 18.) In the figure of § 177, why do f and e intersect?



Proposition 33 (c n)

177. Theorem. *The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides.*



Given: d, e, and f are bisectors of the angles of $\triangle ABC$.

To prove: d, e, and f are concurrent (at O) and $OR = OS = OT$.

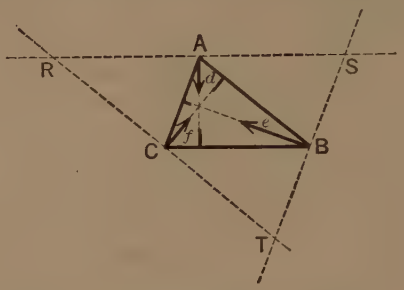
HINT: Use Concurrence Method *B*. What previous theorem relates to the bisector of an angle?

Proof: STATEMENTS	FACTS
1. $\angle A + \angle B + \angle C = 180°$.	1. Why?
2. Since $\angle 1 + \angle 2 \neq 180°$, f and e intersect (at O).	2. *Prin.* 18.
3. Since O is on f, $OR = OT$.	3. § 172.
4. Since O is on e, $OT = OS$.	4. Why?
5. $OR = OT = OS$.	5. Sub.
6. Since $OR = OS$, O is on d.	6. § 173.
7. d, e, and f are concurrent.	7. § 174.

NOTE: Point O is the center of a circle which touches the sides of the triangle. Such a point is often called the *incenter* of the triangle.

Proposition 34 (c n)

178. Theorem. *The altitudes of a triangle are concurrent.*



Given: d, e, and f are altitudes of $\triangle ABC$.

To prove: d, e, and f are concurrent.

HINT: Use Concurrence Method *C*. Show that the altitudes are the perpendicular bisectors of the sides of another triangle.

Proof:

STATEMENTS	FACTS
1. Draw $RAS \parallel CB$, $SBT \parallel AC$, and $RCT \parallel AB$.	1. *Prin.* 14.
2. $RABC$, $ASBC$, and $CABT$ are ▱.	2. Def. of a parallelogram.
3. $RA = CB$; $CB = AS$.	3. § 147.
4. $RA = CB = AS$.	4. Sub.
5. $d \perp CB$.	5. Def. of an altitude.
6. $d \perp RS$.	6. § 99.
7. d is the perpendicular bisector of RS.	7. Def. of a perpendicular bisector.

In like manner, it may be proved that e and f are respectively the perpendicular bisectors of ST and TR.

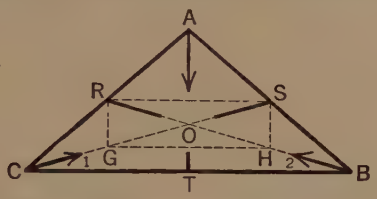
8. d, e, and f are concurrent.	8. § 176.

NOTE: The point of intersection of the altitudes of a triangle is often called the *orthocenter* of the triangle.

NOTE: In an obtuse triangle, the altitudes produced are concurrent.

Proposition 35 (c n)

179. Theorem. *The medians of a triangle are concurrent.*



Given: AT, BR, and CS are medians of $\triangle ABC$.

To prove: AT, BR, and CS are concurrent.

HINT: The proof follows Concurrence Method *A*. (§ 175.)

Proof:

STATEMENTS	FACTS
1. Since $\angle 1 + \angle 2 \neq 180°$, CS and BR intersect (at O).	1. *Prin.* 18.
2. Let G and H be the midpoints of OC and OB.	2. *Prin.* 12.
3. Draw RS, SH, HG, and GR.	3. *Prin.* 13.
4. $RS \parallel CB$ and $GH \parallel CB$.	4. § 155.
5. $RS \parallel GH$.	5. Why?
6. $RS = \frac{1}{2} CB$ and $GH = \frac{1}{2} CB$.	6. Why?
7. $RS = GH$.	7. Why?
8. $RSHG$ is a $\square$.	8. § 152.
9. $CG = OG$.	9. Cons.
10. $OG = OS$.	10. § 149.
11. $CG = OG = OS = \frac{1}{3} CS$. Similarly it may be proved that AT divides CS so that $OS = \frac{1}{3} CS$.	11. Why?
12. Since BR and AT intersect CS so that $OS = \frac{1}{3} CS$, they pass through the same point (O) and are concurrent.	12. Def. of concurrence; § 174.

NOTE: Point O is the *center* of *gravity* of the triangle; it is also often called the *centroid* of the triangle.

180. Trisection Point. If two points divide a line into three equal parts, either point is called a *trisection point* of that line.

181. Corollary (c n). *The medians of a triangle meet at a trisection point of each.*

HINT: Use §§ 179 and 180.

Oral Exercises

1. In what kind of a triangle do the altitudes meet at one of the vertices of the triangle?

2. The incenter of a triangle is equidistant from what?

3. The circumcenter of a triangle is equidistant from what?

4. The orthocenter, the incenter, the circumcenter, and the centroid are all the same point for what kind of a triangle?

5. Criticize: The perpendicular bisectors of the sides of a triangle are concurrent in a point which is equidistant from the sides of the triangle. Is this statement ever true?

6. Is the orthocenter of a triangle always within the triangle?

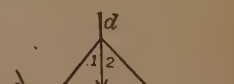
7. Describe the conditions for the circumcenter to be within, on, or without a triangle.

8. What are the two Principles which state the conditions for two lines to intersect?

9. Describe in your own words how three lines m, n, and p, may be proved to be concurrent at a point O.

Exercises. Concurrent Lines

1. In the adjoining figure, if $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, and $\angle 5 = \angle 6$, prove that d, e, and f are concurrent.



HINT: Use Concurrence Method *B*.

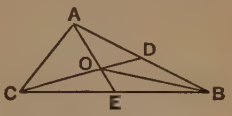
2. Prove that the circumcenter of a right triangle is the mid-point of the hypotenuse.

3. Prove that the bisectors of two angles of an equilateral triangle are concurrent at a trisection point of an altitude.

4. Are the perpendicular bisectors of the sides of a parallelogram ever concurrent? What figure do they form when they are not concurrent? Prove your answer.

5. For what quadrilaterals are the bisectors of the angles concurrent? Prove your answer.

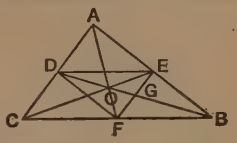
6. In the adjoining figure, CD and EA are straight lines, $AD = DB$, $CE = EB$, $OE = 4$, and $OC = 9$. Find OD and OA.



7. In the figure of Ex. 6, if COD and AOE are medians and if $AC = AE$ and BO is produced through O to meet AC in P, find PA in terms of AE.

8. In an isosceles triangle, the altitude and the bisectors of the base angles are concurrent.

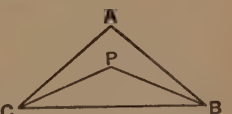
9. In the adjoining figure, AF, CE, and DB are medians. Prove that G is the midpoint of DB.



SUPPLEMENTARY EXERCISES

Exercises. Angles of Triangles

1. In the isosceles triangle ABC, the bisectors of $\angle B$ and $\angle C$ meet at P. How many degrees at A if $\angle P$ is $100°$? $120°$? $112\frac{1}{2}°$? $106\frac{1}{2}°$? $m°$?



2. A and B are the acute angles of a right triangle. Find the number of degrees in each under the following relations:

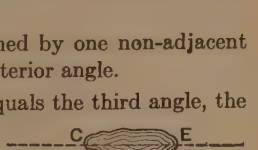
	a	b	c	d	e	f	g
$\angle A$	x	x	x	x	x	x	x
$\angle B$	$2x$	$5x$	$\frac{1}{2}x$	$1\frac{1}{2}x$	$x + 20$	$x - 30$	$.4x$.

3. The angle at the vertex of an isosceles triangle is $70°$. How many degrees are in the angle included between the altitudes drawn from the extremities of the base to the opposite sides?

4. An exterior angle of a triangle diminished by one non-adjacent interior angle equals the other non-adjacent interior angle.

5. If the sum of two angles of a triangle equals the third angle, the triangle is a right triangle.

6. With the aid of the annexed figure explain how one may measure an inaccessible distance such as CE.



7. If the angle at the vertex of an isosceles triangle is 36°, the bisector of a base angle divides the figure into two isosceles triangles.

8. Each base angle of an isosceles triangle is half the supplement of the vertex angle. Prove.

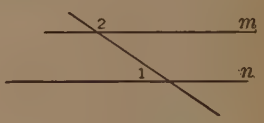
9. If one angle of an isosceles triangle is 60°, the triangle is equiangular.

10. The altitude upon the hypotenuse of a right triangle divides the right angle into parts equal respectively to the acute angles.

11. If the sum of two exterior angles of a triangle equals 270°, the triangle is a right triangle.

Exercises. Parallel Lines

12. In the figure, if $m \parallel n$ and $\angle 1 = 40°$, find all the other angles.



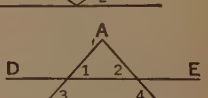
13. In the same figure, if $\angle 1$ and $\angle 2$ are in the ratio of 4:5, find the values of all the angles.

14. In the figure are three equilateral triangles. Prove three pairs of lines parallel.

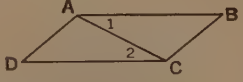
15. If two lines bisect each other, the lines joining their ends are parallel in pairs.

16. In the figure, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$, prove $m \parallel n$.

17. ABC is an isosceles triangle and $DE \parallel CB$. Prove $\angle 1 = \angle 4$.

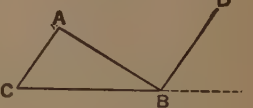


18. In the figure, $AB = DC$ and $\angle 1 = \angle 2$, prove $AD \parallel BC$.



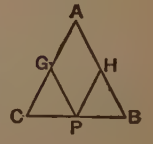
19. If a transversal is perpendicular to two lines, these lines are parallel.

20. In the figure, if $BD \parallel AC$, prove $\angle A + \angle B + \angle C = 180°$.

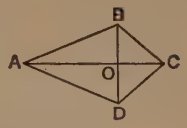


Exercises. Congruent Triangles

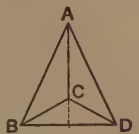
21. In the isosceles triangle ABC, the points G, H, and P bisect the sides. Prove $\triangle GPC \cong \triangle HBP$.



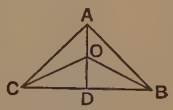
22. In the figure, if BC and DC are equal and form equal angles with AC, prove $\triangle ABO \cong \triangle AOD$.



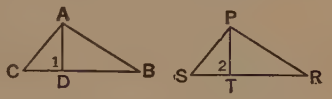
23. In the figure, if AC produced is the perpendicular bisector of BD, prove $\triangle ABC \cong \triangle ACD$.



24. In the figure, if OC and OB are equal and form equal angles with AD, prove $\triangle ADC \cong \triangle ABD$.



25. In the annexed figures, if $AD = PT$, $AC = PS$, $\angle 1 = \angle 2 =$ rt. $\angle$, and $BC = RS$, prove $\triangle ABC \cong \triangle PRS$.



26. In the same figures, if $AD = PT$, $AC = PS$, $\angle 1 = \angle 2 =$ rt. $\angle$, and $AB = PR$, prove $\triangle ABC \cong \triangle PRS$.

27. In the same figures, if $AD = PT$, $AC = PS$, $\angle 1 = \angle 2 =$ rt. $\angle$, and $\angle B = \angle R$, prove $\triangle ABC \cong \triangle PRS$.

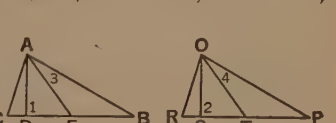
28. In the same figures, if $AD = PT$, $\angle C = \angle S$, $\angle 1 = \angle 2 =$ rt. $\angle$, and $\angle B = \angle R$, prove $\triangle ABC \cong \triangle PRS$.

29. In the same figures, if $AD = PT$, $\angle C = \angle S$. $\angle 1 = \angle 2 =$ rt. $\angle$, and $\angle BAC = \angle RPS$, prove $\triangle ABC \cong \triangle PRS$.

In the annexed figures, $\angle 1 = \angle 2 =$ rt. $\angle$, $CE = EB$, $RT = TP$, $AD = OS$, and $AE = OT$.

30. Prove $\triangle ABC \cong \triangle OPR$, if $AB = OP$.

31. Prove $\triangle ABC \cong \triangle OPR$, if $\angle B = \angle P$.



32. Prove $\triangle ABC \cong \triangle OPR$, if $BC = PR$.

33. Prove $\triangle ABC \cong \triangle OPR$, if $\angle 3 = \angle 4$.

34. Make up an exercise of your own using the above figures. Prove your exercise.

Exercises. Parallelograms

35. The lines drawn from two non-adjacent vertices of a parallelogram to the mid-points of their opposite sides are equal and parallel.

36. The lines drawn from two non-adjacent vertices of a parallelogram to the mid-points of their opposite sides trisect one of the diagonals.

37. The bisectors of opposite angles of a parallelogram which are terminated by a diagonal are equal.

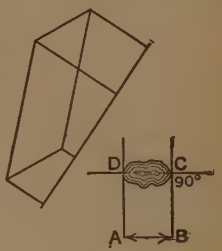
38. The lines joining the mid-points of the adjacent sides of a rectangle form a rhombus.

39. The lines joining the mid-points of the adjacent sides of a rhombus form a rectangle.

40. If the diagonals of a parallelogram are equal, the figure is a rectangle.

41. If one angle of a parallelogram is a right angle, the other three angles are right angles.

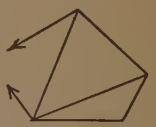
42. If from any point in the base of an isosceles triangle, parallels to the legs are drawn, a parallelogram will be formed whose perimeter equals the sum of the legs.



43. If perpendiculars are drawn from the four vertices of a parallelogram to any line outside the parallelogram, the sum of the perpendiculars from a pair of opposite vertices equals the sum of the perpendiculars from the other pair.

44. With the aid of the annexed figure, show how to measure the length of an inaccessible line such as DC.

Exercises. Angles of Polygons

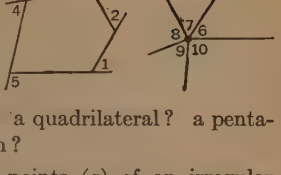


45. The sum of the interior angles of a polygon of n sides equals $(n - 2)$ st. $\measuredangle$. Prove by drawing diagonals from any vertex.

46. How many angles may be right angles in a triangle? in a quadrilateral? in a pentagon? in a hexagon? in an octagon? Why?

47. How many sides has a polygon if the sum of its interior angles equals (*a*) twice the sum of its exterior angles; (*b*) three times the sum of its exterior angles; (*c*) x times the sum of its exterior angles?

48. The sum of the exterior angles of a polygon equals four right angles. Prove by drawing lines which radiate from a point and are parallel to the sides of the polygon.



49. How many diagonals has a triangle? a quadrilateral? a pentagon? a hexagon? an octagon? an n-gon?

50. Find the sum of the angles at the points (a) of an irregular pentagram; (b) of an irregular hexagram.

51. How many sides has a polygon each of whose interior angles equals 135°? 160°? 179°? 120°?

52. How many degrees are in each exterior angle of an equiangular polygon of 9 sides? 10 sides? 40 sides? 100 sides?

53. How many degrees are in each angle of an equiangular triangle? quadrilateral? pentagon? hexagon? octagon? decagon?

54. How many sides has a polygon, the sum of whose interior angles is equal to three times the sum of the angles of a hexagon?

55. In what polygon does the sum of its interior angles equal the sum of its exterior angles?

SUMMARY OF CHAPTER ONE

Triangles are congruent, if they have

(1) *s.a.s.* = *s.a.s.* (3) *s.s.s.* = *s.s.s.*
(2) *a.s.a.* = *a.s.a.* (4) *s.a.a.* = *s.a.a.*

Right triangles are congruent, if they have

(1) *l.l.* = *l.l.* (3) *hy.l.* = *hy.l.*
(2) *l.a.* = *l.a.* (4) *hy.a.* = *hy.a.*

Lines are equal, if they are

(1) corresponding sides of congruent triangles;
(2) legs of an isosceles triangle;
(3) opposite sides of a parallelogram;
(4) equal to a third line or to equal lines;
(5) halves or doubles of the same line or equal lines;
(6) sums or differences of equal pairs of lines;
(7) sides opposite equal angles in the same triangle;
(8) sides of a regular polygon;
(9) perpendiculars to the sides of an angle from a point in the bisector of the angle;

(10) drawn to the extremities of a line from a point in the perpendicular bisector of the line;
(11) segments of intersecting diagonals of a parallelogram;
(12) segments of a side of a triangle formed by a line which bisects a second side and is parallel to a third side;
(13) legs of an isosceles trapezoid.

Angles are equal, if they are

(1) corresponding angles of congruent triangles;
(2) angles opposite equal sides in the same triangle;
(3) opposite angles of a parallelogram;
(4) alternate interior angles of parallel lines;
(5) alternate exterior angles of parallel lines;
(6) exterior-interior angles of parallel lines;
(7) right angles; straight angles; vertical angles;
(8) equal to a third angle or to equal angles;
(9) halves or doubles of the same angle or of equal angles;
(10) complements or supplements of the same angle or of equal angles;
(11) the third angles of two triangles which have their other two angles respectively equal;
(12) angles whose sides are parallel (unless they are supplementary);
(13) angles whose sides are perpendicular (unless supplementary);
(14) angles of a regular polygon.

Lines are parallel, if

(1) they form equal alternate interior angles with the same transversal;
(2) they form equal alternate exterior angles with the same transversal;
(3) they form equal exterior-interior angles with the same transversal;
(4) they form two interior angles on the same side of a common transversal which are supplementary;
(5) they form two exterior angles on the same side of a common transversal which are supplementary;
(6) they are opposite sides of a parallelogram;
(7) they are perpendicular to or parallel to the same line;
(8) one is a side of a triangle and the other bisects the remaining sides;
(9) they are bases of a trapezoid.

In a parallelogram

(1) the opposite sides are parallel;
(2) the opposite sides are equal;
(3) the opposite angles are equal;
(4) two consecutive angles are supplementary;
(5) the diagonals bisect each other;
(6) each diagonal forms two congruent triangles.

A quadrilateral is a parallelogram, if

(1) its opposite sides are parallel;
(2) its opposite sides are equal;
(3) one pair of sides are both equal and parallel;
(4) its diagonals bisect each other;
(5) its opposite angles are equal;
(6) any two pairs of consecutive angles are supplementary.

In any polygon of n sides

(1) the sum of its interior angles equals $(n - 2)\ 180°$;
(2) the sum of its exterior angles equals $360°$.

Lines are concurrent, if they are

(1) perpendicular bisectors of the sides of a triangle;
(2) bisectors of the angles of a triangle;
(3) altitudes of a triangle;
(4) medians of a triangle.

REVIEW ORAL EXERCISES. DIAGNOSTIC TEST

1. Are all adjacent angles equal?

2. What kind of an angle is less than its supplement? equal to its supplement? greater than its supplement?

3. What angle is half of a straight angle? of an obtuse angle?

4. Is the size of an angle decreased by shortening its sides?

5. May two adjacent angles be straight angles?

6. Give four conditions under which triangles are congruent.

7. Give four conditions under which right triangles are congruent.

8. If three bars are hinged together at their ends and so form a triangle, is the frame rigid?

9. If four bars are hinged together at their ends and so form a quadrilateral, is the frame rigid?

10. How may a rectangular frame be made rigid by the addition of another bar?

11. In an obtuse triangle, how many altitudes fall without the triangle?

12. In what kind of a triangle are two medians equal? three medians?

13. In what triangles may an altitude bisect a base?

14. In what three ways may a line be drawn from the vertex of an isosceles triangle to form two congruent triangles?

15. If two angles of a triangle are 40° and 15°, find the third angle.

16. If one angle of a triangle is a right angle, what is the relation between the other two angles?

17. What is the most common method of proving two angles or two lines equal?

18. May the pair of rails of a railroad track be given as an example of parallel lines?

19. When two parallel lines are cut by a transversal what four kinds of equal angles are formed?

20. Are alternate interior angles necessarily equal?

21. What six facts are true of any parallelogram?

22. Under what six conditions will a quadrilateral be a parallelogram?

23. If one angle of a parallelogram is 75°, what are the other angles?

24. What is the sum of the two angles of a trapezoid adjacent to one leg?

25. In what quadrilaterals do the diagonals bisect the angles?

26. In what quadrilaterals are the diagonals equal?

27. In what quadrilaterals are the diagonals perpendicular?

28. What fact may be stated concerning the diagonals of a parallelogram?

29. What two facts may be stated concerning the diagonals of a rectangle?

30. What three facts may be stated concerning the diagonals of a rhombus?

31. What four facts may be stated concerning the diagonals of a square?

32. What is the relation between any interior angle of a polygon and its adjacent exterior angle?

33. What two facts may be stated concerning a line which joins the mid-points of two sides of a triangle?

34. How many points may be equidistant from the extremities of a line? May all of them be located on a certain line?

35. What may be said of all points on the bisector of an angle?

36. What four sets of lines of a triangle are concurrent?

MISCELLANEOUS EXERCISES

1. The bisectors of two consecutive angles of a trapezoid are perpendicular, if they are not adjacent to the same base.

2. If a diagonal of a quadrilateral bisects two of its angles, it is perpendicular to the other diagonal.

3. Triangles ABC and DBC are on the same base BC and on the same side of it. $AB = DC$, $AC = DB$, and AC intersects DB at O. Prove $\triangle BOC$ is isosceles; prove $AD \parallel BC$.

4. If through the vertex of the right angle of a right triangle a line is drawn parallel to the hypotenuse, the legs of the right triangle bisect the angles formed by this parallel and the median drawn to the hypotenuse.

5. If two angles of a triangle are 40° and 50°, find the number of degrees in the angle formed by the bisector of the third angle and the altitude drawn from the same vertex.

6. If the median of a triangle is equal to half the side to which it is drawn, the triangle is a right triangle.

7. If the base of an isosceles triangle is trisected, lines drawn from the vertex to the points of trisection are equal.

8. Two lines parallel respectively to two intersecting lines..... Complete and prove.

9. Every point on the perpendicular bisector of a line segment ..... from the end points of the segment. Complete and prove.

10. The altitude upon one of the legs of an isosceles triangle makes an angle with the base equal to ..... at the vertex. Complete and prove.

11. If the altitude at the mid-point of a side of a triangle is equal to half of that side, what kind of a triangle is it? Prove.

12. The angle between the bisectors of two consecutive angles of a quadrilateral is equal to half the sum of the remaining angles. Prove.

13. If two opposite angles of a quadrilateral are right angles, the bisectors of the other two angles are parallel.

14. If two consecutive angles of a quadrilateral are right angles, the bisectors of the other two angles are perpendicular.

15. The median to the hypotenuse of a right triangle equals half the hypotenuse.

16. The mid-point of the hypotenuse of a right triangle is equidistant from the three vertices.

17. If from the vertices of a square equal distances are measured in succession on the four sides, the figure formed by joining the consecutive points thus obtained is also a square.

18. The lines joining the mid-points of the sides of a triangle form a triangle whose perimeter equals half the perimeter of the original triangle.

19. The bisectors of a pair of vertical angles form a straight line.

20. The bisectors of an angle of a triangle and of the exterior angles at the other two vertices are concurrent.

21. How many sides has a polygon the sum of whose interior angles equals six times the sum of its exterior angles?

22. The line which bisects the opposite sides of a parallelogram is parallel to the other two sides and equal to half their sum.

23. If from the point where the bisector of an angle of a triangle meets the opposite side, parallels are drawn to the other two sides, a rhombus is formed.

24. From the vertex and the mid-points of the equal sides of an isosceles triangle perpendiculars are drawn to the base; prove that they divide the base into four equal parts.

25. If one base of a trapezoid is double the other, the diagonals intersect at a trisection point of each.

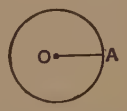
CHAPTER TWO

CIRCLES—THEIR LINES AND ANGLES

CIRCLES AND THEIR LINES

182. Circle ($\odot$). A *circle* is a closed curve all points of which are equidistant from a point within called the *center*. The pupil should notice that a circle is defined as *a line*, not as a portion of a plane.

183. Radius (plural, **radii**). A *radius* is a straight line from the center to any point on the circle. *OA* is a radius.

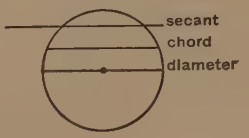


184. Equal Circles. Circles are equal (or congruent) if their radii are equal.

185. Concentric Circles. Circles having the same center are *concentric* circles.

186. Secant. A *secant* is a straight line which intersects a circle in two points.

187. Chord. A *chord* is a straight line which joins any two points on a circle.



188. Diameter. A *diameter* is a chord which passes through the center.

Oral Exercises

1. What may be said concerning the lengths of all radii of a circle?
2. How may equal circles be made to coincide?
3. What is the difference between a secant and a chord?

4. What is the relation of a diameter to a radius?

5. Why do two diameters of a circle bisect each other?

6. What is the longest chord of a circle?

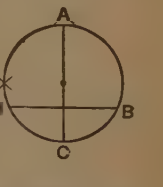
7. From any point on a circle, how many chords may be drawn? how many diameters? how many radii?

8. Through any point within a circle, how many chords may be drawn? how many diameters? how many radii?

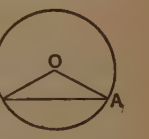
9. What lines of equal circles are equal?

10. How many circles may be drawn having the same center? having equal radii? having the same center and different radii? having the same center and a given radius?

189. Arcs. An *arc* is a part of a circle. The circle in the figure is divided into five parts or arcs. The symbol for arc is $\frown$. Arc AB is written $\overset{\frown}{AB}$ and read as "arc AB." A *semicircle* is an arc which is half of a circle. In the figure, if $\overset{\frown}{ABC} = \overset{\frown}{ADC}$, each arc is a semicircle. A *quadrant* is one-fourth of a circle. If E is the mid-point of $\overset{\frown}{AC}$, $\overset{\frown}{AE}$ and $\overset{\frown}{EC}$ are quadrants. A *minor arc* is less than a semicircle; a *major arc* is greater than a semicircle. In the figure, $\overset{\frown}{BCD}$ is a minor arc; $\overset{\frown}{DAB}$ is a major arc. When the letters used to name an arc may stand for two arcs, the minor arc is meant. For example, $\overset{\frown}{BD}$ means the minor arc BCD.



190. Central Angles. A *central angle* is an angle formed by two radii, as $\angle O$. If the sides of the angle intercept $\overset{\frown}{AB}$, $\angle O$ is said to have $\overset{\frown}{AB}$ as its arc and $\overset{\frown}{AB}$ has $\angle O$ as its angle. Likewise chord AB is said to have $\overset{\frown}{AB}$ as its arc and $\overset{\frown}{AB}$ has AB as its chord.



191. Circumference. The *length* of a circle is called its *circumference*. It is measured in the usual *linear units* and also in a new unit of length called an *arc degree*. An arc of

$1°$ means $\frac{1}{360}$ of the circumference. In navigation an arc of one minute ($1'$) measured on a great circle of the earth is called a *nautical mile*. *An arc degree is the unit used to measure the length of an arc* and it should not be confused with an *angle degree* which is used to measure the size of an angle. Principle 31 states that *a central angle equals numerically its intercepted arc.* This means that in the case of a central angle, the ***number** of angle degrees* equals the ***number** of arc degrees* in its arc.

Oral Exercises

1. How many arcs does each chord of a circle have? What is the sum of those arcs?

2. If a chord divides a circle into equal arcs, what is the chord called?

3. What is the straight line called which joins the extremities of an arc?

4. What is meant by the statement that a ship is in latitude 60° North?

5. If the circumference of the earth measured on a great circle is taken as 25,000 statute miles, how would you find the length of a nautical mile in terms of statute miles?

6. How many degrees in the arc intercepted by a central angle of 90°? 74°? $m°$?

192. Ratio. The *ratio* of two quantities of the same kind is the *quotient of their numerical measures in terms of a common unit.* The ratio of 3 quantities to 4 quantities of the same kind may be written in the fractional form $\frac{3}{4}$ or in the form $3:4$, and either may be read, "3 to 4" or "3 is to 4." If $\angle A = 30°$ and $\angle B = 40°$, $\frac{\angle A}{\angle B} = \frac{30°}{40°} = \frac{3}{4}$, which is read, "The ratio of $\angle A$ to $\angle B$ equals the ratio of 3 to 4", or "$\angle A$ is to $\angle B$ *as* 3 is to 4."

If two quantities are equal, their ratio is equal to one or

unity. For example, if $a = b$, $\frac{a}{b} = 1$, (unity), by the Axiom of Division.

If two quantities have the ratio of unity, they are equal. For example, if $\frac{a}{b} = 1$, $a = b$ by the Axiom of Multiplication.

Principle 2, p. 396, states that *two magnitudes are equal if their ratio is unity; and conversely.*

Oral Exercises

1. Which of the four fundamental operations is used in finding a ratio?

2. If A is 75% of B, what is their ratio?

3. Give the following ratios in their simplest form: (*a*) 36:72; (*b*) 65:78; (*c*) \$16 : \$20; (*d*) 1.5 : 3.5; (*e*) $1\frac{1}{2} : 2\frac{1}{2}$; (*f*) $2\ AB : 3\ AB$; (*g*) $4\ AB : 4\ XY$; (*h*) $2\angle B : m\angle B$.

4. What is the ratio of 2 ft. to 2 in.? Of 2 ft. to 2 yd.?

5. What is the ratio of 2 qt. to 2 pt.? Of 2 qt. to 2 gal.?

6. If $\angle A = 35°$ and $\angle B = 35°$, what is their ratio? If any two quantities are equal what is their ratio?

7. Is it possible to express the ratio of 2 ft. to 2 qt.? Give a reason for your answer.

8. If $\angle A = 60°$ and arc $x = 60°$, are they equal? Are they numerically equal?

9. If $\overset{\frown}{m}$ is one-half of $\overset{\frown}{n}$ and their sum is 180°, find the number of degrees in each.

HINT: $2m = n$ and $m + n = 180°$.

10. If $\overset{\frown}{m}$ and $\overset{\frown}{n}$ are respectively the minor and the major arcs of the same chord, find the number of degrees in each if their ratio is (*a*) 1:2; (*b*) 2:3; (*c*) 4:5.

Principles of Chapter Two

The following new principles will be used in Chapter Two:

No. 2. Two magnitudes are equal if their ratio is unity; and conversely.

No. 21. A straight line intersects a circle in at most two points.

No. 31. A central angle equals numerically its intercepted arc.

No. 32. One circle and only one can be drawn with any point as a center and with a radius of any given length.

No. 33. Radii of the same circle (or of equal circles) are equal.

No. 34. A point is within, on, or without a circle according as its distance from the center is less than, equal to, or greater than a radius.

No. 35. Two circles are equal if their radii are equal.

No. 36. A diameter bisects a circle.

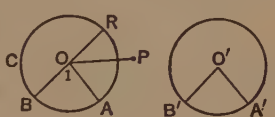
Oral Exercises

1. If $x = y$, to what is the ratio of x to y equal? The ratio of $2x$ to $2y$ is equal to what?

2. May a straight line intersect a circle in one point? two points? three points?

3. If $\overset{\frown}{AB} = 60°$, why does $\angle 1 = 60°$?

4. Why does OA equal OB?



5. Are five dimes and five quarters equal? Are they equal numerically?

6. If circle O and circle O' are equal, why does $OA = O'A'$?

7. If the circumference of circle O' is 720 mm., how long is an arc of 5° (arc degrees)?

8. A central angle of 60° (angle degrees) in a circle whose circumference is x feet has an arc of how many degrees (arc degrees)? Of how many feet?

9. If OP is greater than OA, why is point P without the circle? (The term *without* in geometry may mean "outside.")

10. If OR equals radius OA, why is point R on the circle?

11. If $OA = O'A'$, why does circle O = circle O'?

12. If BOR is a diameter, why does $\overset{\frown}{BAR}$ equal $\overset{\frown}{BCR}$?

13. In the figure of Proposition 1, which follows, if $\angle O = 60°$ and $\angle O' = 20°$, what is the number of degrees in $\overset{\frown}{x}$ and $\overset{\frown}{x'}$? $\angle O$ is how many times $\angle O'$? $\overset{\frown}{x}$ is how many times $\overset{\frown}{x'}$?

Proposition 1 (c N)

193. Theorem. *Central angles of a circle (or of equal circles) have the same ratio as their arcs.*

Given: Central $\measuredangle O$ and O' with $\overset{\frown}{x}$ and $\overset{\frown}{x'}$ in equal circles.

To prove: $\dfrac{\angle O}{\angle O'} = \dfrac{\overset{\frown}{x}}{\overset{\frown}{x'}}$.

HINT: What principle relates to central angles? What operation is indicated by a ratio?

Proof: STATEMENTS	FACTS
1. $\angle O$ = numerically $\overset{\frown}{x}$, and $\angle O'$ = numerically $\overset{\frown}{x'}$.	1. *Prin.* 31.
2. $\dfrac{\angle O}{\angle O'} = \dfrac{\overset{\frown}{x}}{\overset{\frown}{x'}}$.	2. Axiom of Division.

NOTE: The conclusion of section 193 is an equation which states that a certain relationship (the ratio) between two central angles is the same as a like relationship between the arcs of those angles. It should be remembered that this equation is taken for the numerical measures of the four quantities involved. It means for example, if the number of angle degrees in $\angle O$ is three times the number of angle degrees in $\angle O'$, then the number of arc degrees in $\overset{\frown}{x}$ is three times the number of arc degrees in $\overset{\frown}{x'}$.

Proposition 2 (c N)

194. Theorem. *Equal central angles of a circle (or of equal circles) have equal arcs; and, equal arcs have equal central angles.*

Given (1): $\angle O = \angle O'$ with $\overset{\frown}{x}$ and $\overset{\frown}{x'}$ in equal circles.

To prove (1): $\overset{\frown}{x} = \overset{\frown}{x'}$.

HINT: Two central angles have what ratio? Equal angles have what ratio?

Proof (1):

STATEMENTS	FACTS
1. $\frac{\angle O}{\angle O'} = \frac{\overset{\frown}{x}}{\overset{\frown}{x'}}$.	1. § 193.
2. $\frac{\angle O}{\angle O'} = 1$.	2. Given and *Prin.* 2.
3. $\frac{\overset{\frown}{x}}{\overset{\frown}{x'}} = \frac{\angle O}{\angle O'} = 1$.	3. Sub.
4. $\overset{\frown}{x} = \overset{\frown}{x'}$.	4. *Prin.* 2.

Second Part of Proposition 2

Given (2): $\overset{\frown}{x} = \overset{\frown}{x'}$ with central $\measuredangle O$ and O' in equal circles.

To prove (2): $\angle O = \angle O'$.

HINT: Two arcs have what ratio? Equal arcs have what ratio?

Proof (2): (The pupil may complete the demonstration.)

195. Converse Theorems. The second part of Proposition 2 is called the *converse* of the first part. The converse of a theorem is formed by interchanging significant statements of the *hypothesis* and the *conclusion.* The original theorem may be called the *direct* theorem.

The following illustrations will show how the hypothesis and the conclusion of some of the theorems of Chapter One are interchanged:

Direct Theorem. (§ 172.)

Given: If a point is on the bisector of an angle,

To prove: the point is equidistant from the sides of the angle.

Converse Theorem. (§ 173.)

Given: If a point is equidistant from the sides of an angle,

To prove: the point is on the bisector of the angle.

Direct Theorem. (§ 79.)

Given: If a triangle is isosceles,

To prove: the base angles of the triangle are equal.

Converse Theorem. (§ 120.)

Given: If the base angles of a triangle are equal,

To prove: the triangle is isosceles.

Oral Exercises

The converse of a theorem is not always true. State the converses of the following theorems and determine, if you can, whether they are true or false:

1. A point on the perpendicular bisector of a line is equidistant from the extremities of the line.

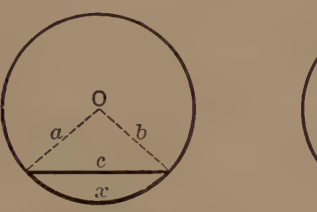
2. A line which bisects two sides of a triangle is parallel to the third side.

3. Alternate interior angles of parallel lines are equal.

4. A diagonal divides a parallelogram into two congruent triangles.

Proposition 3 (c N)

196. Theorem. *Equal arcs of a circle (or of equal circles) have equal chords; conversely, equal chords have equal arcs.*



Given: $\overset{\frown}{x} = \overset{\frown}{x'}$ with chords c and c' in equal circles.
To prove: $c = c'$.

HINT: What is the usual method of proving lines equal? How may central angles be proved equal?

Proof: STATEMENTS	FACTS
1. Draw radii a, b, a' and b'. In $\triangle abc$ and $\triangle a'b'c'$,	1. *Prin.* 13.
2. $a = a'$ and $b = b'$.	2. *Prin.* 33.
3. $\overset{\frown}{x} = \overset{\frown}{x'}$.	3. Given.
4. $\angle O = \angle O'$.	4. Why?
5. $\triangle abc \cong \triangle a'b'c'$.	5. Why?
6. $c = c'$.	6. Why?

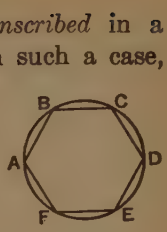
Conversely

Given: $c = c'$ with $\overset{\frown}{x}$ and $\overset{\frown}{x'}$ in equal circles.

To prove: $\overset{\frown}{x} = \overset{\frown}{x'}$.

Proof: STATEMENTS	FACTS
1.	1. .....
2.	2. .....
3.	3. .....
4. $\triangle abc \cong \triangle a'b'c'$.	4. *s.s.s.* = *s.s.s.*
5.	5. .....
6. $\overset{\frown}{x} = \overset{\frown}{x'}$.	6. .....

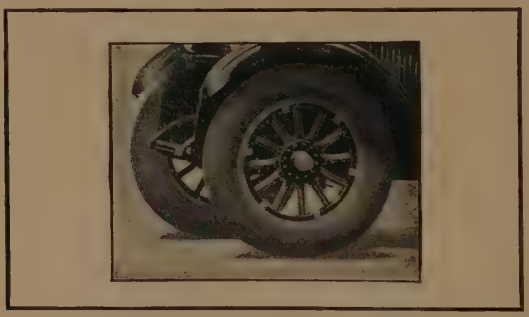
197. Inscribed Polygon. A polygon is *inscribed* in a circle if its sides are chords of the circle. In such a case, the circle is said to be *circumscribed* about the polygon and the center of the circle is called the *circumcenter* of the polygon.



ABCDEF is an inscribed polygon.

198. Tool XII. Arcs are equal if their $\left\{\begin{matrix}\text{central angles}\\ \text{chords}\end{matrix}\right\}$ are equal.

199. Tool XIII. $\left\{\begin{matrix}\text{Central angles}\\ \text{Chords}\end{matrix}\right\}$ are equal if their arcs are equal.



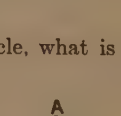
EQUAL ARCS AND EQUAL ANGLES

What angle is formed by two adjacent spokes? How many arc degrees are between two adjacent rim lug-nuts? What other illustrations of circles can you find in the picture?

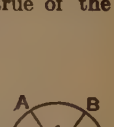
Oral Exercises

1. If a regular polygon is inscribed in a circle, what is true of the arcs? Why?

2. If the sides of an inscribed polygon have equal arcs, what is true of the polygon? Why?



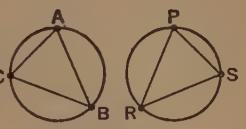
3. If arc AB = arc AC, why is the triangle isosceles?



4. If AOC and BOD are diameters, why does $\overset{\frown}{AB} = \overset{\frown}{CD}$?

5. If $\overset{\frown}{AB} = \overset{\frown}{CD}$ and $\overset{\frown}{BC} = \overset{\frown}{AD}$, why is $ABCD$ a parallelogram?

6. If the triangles ABC and PRS are inscribed in equal circles and if $\overset{\frown}{AC} = \overset{\frown}{PS}$, $\overset{\frown}{AB} = \overset{\frown}{PR}$, and $\overset{\frown}{BC} = \overset{\frown}{RS}$, why are the triangles congruent?



7. If a central angle is doubled, is its arc doubled? Give a reason for your answer.

8. If a central angle is doubled, is its chord doubled?

9. If an arc is doubled, is its chord doubled? Determine your answer by several trial measurements.

10. How many *arc degrees* are in each arc of a circle circumscribed about an equilateral triangle? About a square?

Exercises. Central Angles, Arcs, and Chords

1. Two pairs of equal arcs are made by two diameters of a circle.

2. If O is the center of a circle and $\angle 1 = \angle 2$, prove $\overset{\frown}{AD} = \overset{\frown}{CB}$.

3. *Conversely*, if O is the center and $\overset{\frown}{AD} = \overset{\frown}{CB}$, then $\angle 1 = \angle 2$.

4. If the line joining the mid-points of two radii of a circle equals the line joining the mid-points of two other radii of that circle, the radii intercept equal arcs.

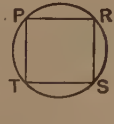
5. If OD is a radius and $AC = CB$, prove $\overset{\frown}{AD} = \overset{\frown}{DB}$.

HINT: Draw radii OA and OB.

6. If $\overset{\frown}{PR} = \overset{\frown}{PS} = \overset{\frown}{RS}$, prove $\triangle PRS$ equiangular.

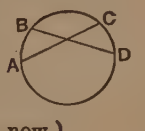
7. If diameters AC and BD are perpendicular, $ABCD$ is a square.

8. If $PRST$ is a square, the circle is divided into quadrants.



9. If $\overset{\frown}{AB} = \overset{\frown}{CD}$, chord AC = chord BD.

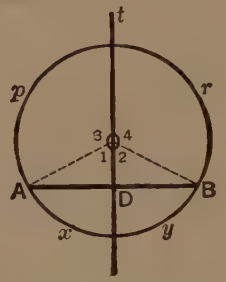
10. *Conversely*, if chord AC = chord BD then $\overset{\frown}{AB} = \overset{\frown}{CD}$.



(Supplementary Exercises 1–12, p. 159, may be studied now.)

Proposition 4 (C N)

200. Theorem. *A line through the center of a circle perpendicular to a chord bisects the chord and its two arcs.*



Given: $t \perp AB$ at D and passes through center O.

To prove: $AD = DB$, $\overset{\frown}{x} = \overset{\frown}{y}$, $\overset{\frown}{p} = \overset{\frown}{r}$.

HINT: How are line segments proved equal? How are arcs proved equal?

Proof: STATEMENTS	FACTS
1. Draw radii OA and OB. In rt. $\triangle ODA$ and rt. $\triangle ODB$,	1. *Prin.* 13.
2. OA = ?	2. Why?
3. OD = ?	3. Why?
4. rt. $\triangle ODA \cong$ rt. $\triangle ODB$.	4. *hy.l.* = *hy.l.*
5. $AD = DB$.	5. Why?
6. $\angle 1 = \angle 2$.	6. Why?
7. $\overset{\frown}{x} = \overset{\frown}{y}$.	7. Why?
8. $\angle 3$ is the supp. of $\angle 1$. $\angle 4$ is the supp. of $\angle 2$.	8. Why?
9. $\angle 3 = \angle 4$.	9. Why?
10. $\overset{\frown}{p} = \overset{\frown}{r}$.	10. Why?

201. Corollary 1 (c N). *A line through the center that bisects a chord, which is not a diameter, is perpendicular to the chord.*

Figure: Use the figure of § 200.

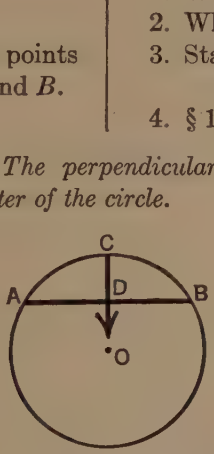
Given: t is a line through center O making $AD = DB$.

To prove: $t \perp AB$.

HINT: What two conditions determine the perpendicular bisector of a line?

Proof: STATEMENTS	FACTS
1. $OA = OB$.	1. Why?
2. $AD = DB$.	2. Why?
3. O and D are points equidistant from A and B.	3. Statements 1 and 2.
4. $t \perp AB$.	4. § 171.

202. Corollary 2. *The perpendicular bisector of a chord passes through the center of the circle.*



Given: $CD \perp AB$, $AD = DB$ in circle O.

To prove: CD produced passes through center O.

HINT: Since CD is the perpendicular bisector of AB, and $OA = OB$, O lies on CD. (§ 170.)

(The pupil should arrange the proof.)

Oral Exercises

1. May § 200 be stated, "A diameter perpendicular to a chord bisects the chord and its two arcs"? What is the advantage of stating

it as in § 200? If the word *radius* is substituted for the word *diameter* what further change is necessary?

2. What is true of the perpendicular bisectors of the sides of an inscribed polygon?

3. If AD, the altitude of an inscribed isosceles triangle, is produced to E, show that AE is a diameter.

Exercises. Bisected Chords and Arcs

1. A line from the mid-point of a chord to the mid-point of its arc is perpendicular to the chord.

2. If a radius bisects an arc, it bisects the chord of the arc and is perpendicular to the chord.

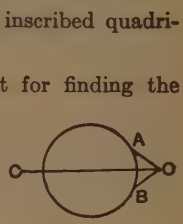
3. If points A and B on a circle are equidistant from the extremities of a chord, show that AB is a diameter.

4. If chords AB and BC are equal, prove that radius OB bisects the chord AC.

5. *Conversely*, if radius OB bisects chord AC, $AB = BC$.

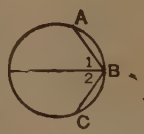
6. The perpendicular bisectors of the sides of an inscribed quadrilateral meet in a point.

7. The annexed figure represents an instrument for finding the center of a circular plate or section of a shaft. The rod OC bisects $\angle AOB$ and $OA = OB$. Prove that if A and B rest on the circle, OC passes through the center. Show how to locate the center.



8. A line which bisects the two arcs of a chord is the perpendicular bisector of the chord.

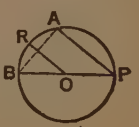
9. If AB and BC are equal chords of the circle, prove that the bisector of their included angle is a diameter.



10. If AB and CD are parallel chords and diameter GH bisects CD, prove that GH bisects AB.

11. In the figure, O is the center of the circle and $OR \parallel AP$. Prove that $\overset{\frown}{AB}$ is bisected by R.

HINT: Use § 154.



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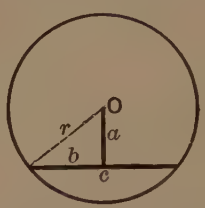
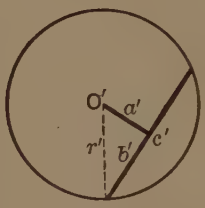
STAR TRAILS

Star trails photographed with a stationary camera form a marvelous illustration of concentric circles. (From a photograph.) Plato said: "God eternally geometrizes." The illustration shows that the North Star is not exactly above the North Pole. The trail of the North Star is represented by the heavy white arc at the left of the center. The circles are incomplete because of the interference of daylight.

Proposition 5 (C N)

203. Theorem. *Equal chords of a circle (or of equal circles) are equally distant from the center; conversely, chords equally distant from the center are equal.*





Given: Circle O = circle O', $c = c'$, $a \perp c$, $a' \perp c'$.

To prove: $a = a'$.

HINT: How is the distance from a point to a line measured?

Proof:

STATEMENTS	FACTS
1. Draw radii r and r'.	1. *Prin.* 13.
2. $a \perp c$ and $a' \perp c'$.	2. Given.
In rt. $\triangle$ arb and rt. $\triangle$ $a'r'b'$,	
3. $r = r'$.	3. Why?
4. $b = \frac{1}{2}\, c$ and $b' = \frac{1}{2}\, c'$.	4. § 200.
5. $c = c'$.	5. Why?
6. b (or $\frac{1}{2}\, c$) = b' (or $\frac{1}{2}\, c'$).	6. Why?
7. rt. $\triangle$ $arb \cong$ rt. $\triangle$ $a'r'b'$.	7. Why?
8. $a = a'$.	8. Why?

Conversely

Given: Circle O = circle O', $a \perp c$, $a' \perp c'$, $a = a'$.

To prove: $c = c'$.

Figure: (As in the direct theorem.)

HINT: If the halves of two lines are equal, are the lines equal?

Proof:

STATEMENTS	FACTS
1. Draw radii r and r'.	1. *Prin.* 13.
2. $a \perp c$ and $a' \perp c'$.	2. Given.
In rt. $\triangle arb$ and rt. $\triangle a'r'b'$,	
3. $r = ?$	3. Why?
4. $a = ?$	4. Why?
5. rt. $\triangle arb \cong$ rt. $\triangle a'r'b'$.	5. Why?
6. $b = b'$.	6. Why?
7. $c = 2b$ and $c' = 2b'$.	7. Why?
8. c (or $2b$) $= c'$ (or $2b'$).	8. Why?

204. Tool XIV. Chords of a circle are equal if the perpendiculars drawn to them from the center are equal.

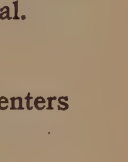
Exercises. Chords and Their Distances From Centers

(Use Tool XIV.)

1. Two chords which are equally distant from the center have equal arcs.

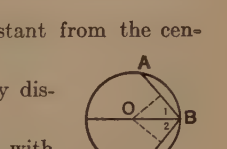
2. Two chords with equal arcs are equally distant from the center.

3. The sides of an inscribed square are equally distant from the center.



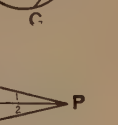
4. Two chords AB and BC make equal angles with the diameter through B. Prove the chords equal.

5. Perpendiculars from the center of a circle to the sides of an inscribed equilateral triangle are equal.



6. In the figure, $\angle 1 = \angle 2$ and O is the center. Prove $AB = CD$.

7. *Conversely*, if O is the center and $AB = CD$, then $\angle 1 = \angle 2$.

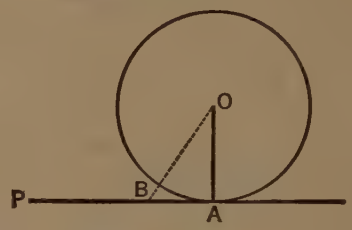


8. If two chords intersect a diameter at P and make equal angles with it, the chords are equal.

(Supplementary Exercises 13–19, p. 160, may be studied now.)

Proposition 6 (c N)

205. Theorem. *A line perpendicular to a radius at its outer extremity has only one point in common with the circle.*



Given: $PA \perp OA$, a radius of circle O.

To prove: A is the only point common to PA and the circle.

HINT: Assume that another point is common to the circle and the line. (See § 115). Use *Prin.* 15.

Proof:

STATEMENTS	FACTS
Let us assume that B is any other point than A common to PA *and the circle.*	
1. Draw radius OB.	1. Why?
2. $OB = OA$.	2. Why?
3. $\angle B = \angle A$.	3. Why?
4. $PA \perp OA$ (that is $\angle A = 90°$).	4. Why?
5. $\angle B = \angle A = 90°$.	5. Why?
6. OB (as well as OA) is perpendicular to PA from O.	6. Why?
7. But OB and OA cannot both be perpendicular to PA.	7. *Prin.* 15.

Since the assumption that there can be another common point (B) leads to an absurdity, the assumption is false and therefore PA has only one point (A) in common with the circle.

Courtesy International Newsreel

CIRCLES IN LANDSCAPES

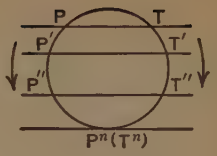
Circles and their lines are used extensively in landscape architecture. The artistic appeal of the circle is due to its perfect symmetry.

TANGENTS

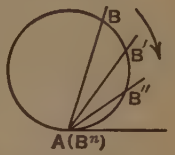
206. Tangent. A straight line that has only one point in common with a circle is called a *tangent* to the circle; the circle is also said to be tangent to the line. Tangent is derived from the Latin verb *tango* meaning "to touch," hence literally a tangent is a line which *touches* a circle at a point. The point common to the tangent and the circle is called the *point of tangency* or the *point of contact.*

In more advanced mathematics, a tangent is defined as the *limiting position of a variable secant* as one point of intersection approaches and coincides with the other point of intersection. In such a conception there are two points which mark the same place of tangency—one point upon the other.

If the secant PT moves so that each succeeding position is parallel to the original position, the two points of intersection P and T approach each other until eventually the two points as shown in the figure become the one point $P^n(T^n)$ and the secant becomes a tangent.

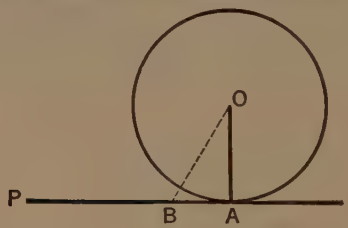


If the secant AB rotates about one point of intersection (A), the other point of intersection (B) approaches A and finally rests upon it in the double point $A(B^2)$ and the secant becomes a tangent.



207. Corollary 1 (c N). *A line perpendicular to a radius at its outer extremity is a tangent to the circle.* (By § 205 and Def.)

208. Corollary 2 (c N). *A tangent and a radius drawn to the point of tangency are perpendicular to each other.*



Given: PA is tangent to circle O at A; OA is a radius.

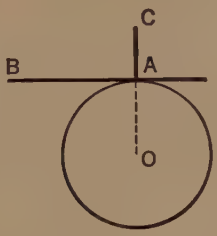
To prove: $OA \perp PA$.

HINT: What kind of line is the shortest line between a point and a line?

Proof:

STATEMENTS	FACTS
1. Point A is the only point common to the tangent and the circle.	1. Given and definition.
2. From any other point B on the tangent, draw BO.	2. *Prin.* 13.
3. Since B is without the circle, OB is longer than a radius (OA).	3. *Prin.* 34.
4. OA is the shortest line from O to PA, therefore $OA \perp PA$.	4. *Prin.* 23.

209. Corollary 3. *A perpendicular to a tangent at the point of tangency passes through the center of the circle.*



Given: BA is tangent to circle O at A; $CA \perp BA$.

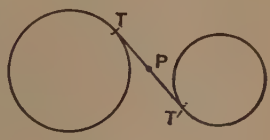
To prove: CA passes through center O.

HINT: A radius is perpendicular to the tangent.

Proof: STATEMENTS	FACTS
1. Draw radius OA.	1. By what Principle?
2. $OA \perp BA$.	2. Why?
3. $CA \perp BA$.	3. Why?
4. CAO is a straight line.	4. *Prin.* 15.

NOTE: Corollaries 1, 2, and 3 involve two lines meeting at a common point on a circle (the *radius* and the *tangent*) and a relationship between those lines (that they are *perpendicular* to each other). If any two of these three terms are true in a figure, the third is also true.

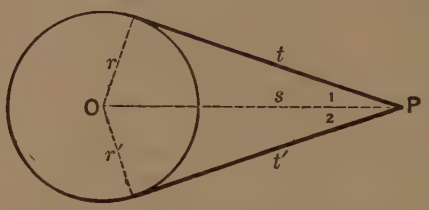
210. A Tangent from an External Point. A tangent from an external point to a circle is the line segment between the external point and the point of tangency.



PT and PT' are tangents from P.

Proposition 7 (c n)

211. Theorem. *Tangents from a point to a circle are equal and make equal angles with the line from the point to the center.*



Given: t and t' are tangents from P to circle O.

To prove: $t = t'$ and $\angle 1 = \angle 2$.

HINT: How are line segments usually proved equal? What angle is formed by a tangent and a radius at the point of tangency?

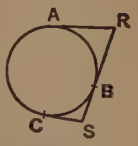
Proof: STATEMENTS	FACTS
1. Draw s from P to O and draw radii r and r' to the points of tangency.	1. By what Principle?
2. $r \perp t$ and $r' \perp t'$.	2. Why?
In rt. $\triangle rst$ and rt. $\triangle r's t'$,	
3. $r = ?$	3. Why?
4. $s = ?$	4. Why?
5. rt. $\triangle rst \cong$ rt. $\triangle r's t'$.	5. Why?
6. $t = t'$ and $\angle 1 = \angle 2$.	6. Why?

212. Circumscribed Polygon. A polygon is *circumscribed about* a circle if its sides are tangent to the circle. In such a case the circle is said to be *inscribed in* the polygon and its center is called the *incenter* of the polygon.

The annexed figures illustrate circumscribed polygons and inscribed circles.

213. Tool XV. Lines are equal if they are tangents from a point to a circle.

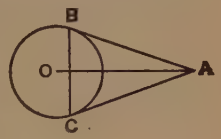
Exercises. Tangents



1. Tangents to a circle at A, B, and C meet at R and S. Show that $AR + CS = RS$.

2. A circumscribed parallelogram is equilateral.

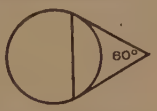
3. If a quadrilateral is circumscribed about a circle, the sum of one pair of opposite sides equals the sum of the other pair.



4. If AB and AC are tangents to a circle whose center is O, AO is the perpendicular bisector of BC.

5. A chord parallel to a tangent is bisected by the diameter drawn from the point of contact.

6. If two circles are concentric, tangents to the inner circle which are limited by the outer circle are equal.



7. If two tangents form a 60° angle, the chord joining the points of contact equals the tangents.

8. If two tangents from an exterior point form an angle of 120°, the distance from the point to the center equals the sum of the tangents.

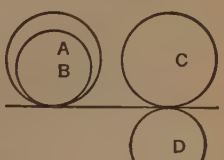
9. Tangents at the extremities of a diameter are parallel.

10. *Conversely*, if two tangents to a circle are parallel, they lie at opposite extremities of a diameter.

11. All chords which are parallel to a tangent are bisected by a diameter drawn to the point of contact.

(Supplementary Exercises 20–30, p. 160, may be studied now.)

214. Tangent Circles. Circles tangent to a line at the same point are called *tangent circles*. If the circles lie on the same side of the tangent line, they are tangent *internally*, as circles A and B. If the circles lie on opposite sides of the line, they are tangent *externally*, as circles C and D.



215. Theorem A. *The line joining the centers of two intersecting circles is the perpendicular bisector of their common chord.*

HINT: What points determine the perpendicular bisector of a line? How is each center related to the ends of the common chord?

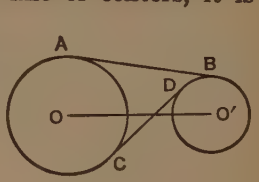
Theorem B. *The line joining the centers of two tangent circles passes through the point of contact.*

HINT: The perpendicular to the common tangent at the point of contact passes through each center. (209). This line coincides with the line joining the centers. (Prin. 13).

NOTE: The line joining the centers of two circles is often called the *line of centers* of the circles.

NOTE: The proofs of Theorems A and B are not required by certain syllabi. The theorems are useful, however, in proving exercises.

216. Common Tangents. A line tangent to each of two circles is called a *common tangent.* If a tangent to two circles intersects the line of centers, it is called a *common internal tangent.* If a tangent to two circles intersects the line of centers produced or is parallel to the line of centers, it is called a *common external tangent.* The length of a common tangent is the length of the segment between the points of contact.



AB is a common external tangent.

CD is a common internal tangent.

Oral Exercises

1. How many circles may have only one point in common?

2. How many circles may have only two points in common? How are they related?

3. How many equal circles may be tangent to a line?

4. How many equal circles may be tangent to a line at a given point?

5. How many common internal tangents may two circles have if they (*a*) are tangent; (*b*) intersect each other; (*c*) are concentric?

6. How many common external tangents may two circles have if they (*a*) are tangent; (*b*) intersect each other; (*c*) are concentric?

7. How many common chords may two circles have if they (*a*) are tangent? (*b*) intersect each other? (*c*) are concentric?

8. How does the line of centers of two circles compare with the sum of the radii, if the circles (*a*) lie apart? (*b*) are tangent? (*c*) intersect each other?

9. Find the length of the line of centers of two circles which are tangent externally and whose diameters are 6 and 4.

10. Find the length of the line of centers of two circles which are tangent internally and whose diameters are 10 and 4.

11. Three circles whose diameters are 4, 6, and 8 are tangent externally, each to the other two. Find the perimeter of the triangle formed by their lines of centers.

Exercises. Common Tangents

1. Sketch two circles with two common internal tangents and two common external tangents.

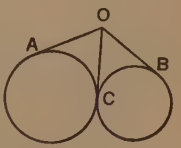
2. Sketch two circles with one common internal tangent and two common external tangents.

3. Sketch two circles with no common internal tangent and two common external tangents.

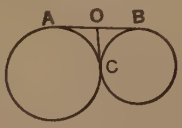
4. Sketch two circles with no common internal tangent and one common external tangent.

5. Sketch two circles with no common tangent.

6. If two circles are tangent externally, the tangents to them from any point of the common internal tangent are equal.

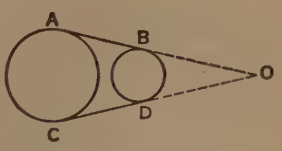


7. If two circles are tangent externally, their common internal tangent bisects either common external tangent between the points of tangency.

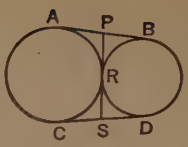


8. The common external tangents to two circles are equal.

9. The common internal tangents to two circles are equal.

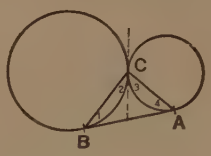


10. The two common external tangents to two tangent circles intercept on the common internal tangent a segment equal to either external tangent.



11. If two circles are tangent at C and if AB is a common external tangent, the angle ACB is a right angle.

HINT: $\angle 1 + \angle 2 + \angle 3 + \angle 4 = ?$



Oral Exercises

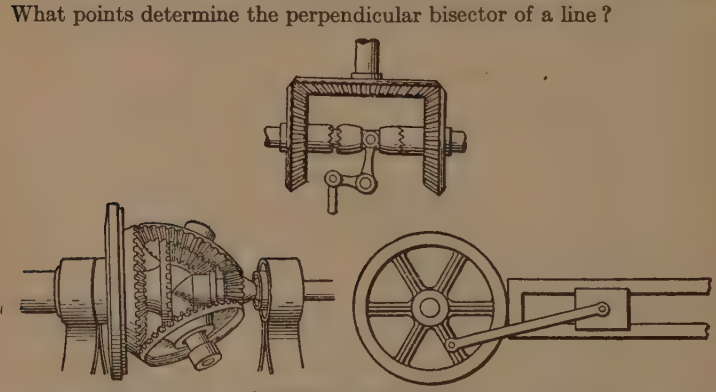
1. How many points may be equidistant from a given point? Upon what kind of a line do all such points lie?

2. How many points may be equidistant from two given points? Upon what kind of a line do all such points lie? What theorem of Chapter One relates to this condition?

3. How many points may be equidistant from the three vertices of a triangle? What theorem of Chapter One relates to this condition?

4. How many circles may be drawn through a point? How many circles may be drawn through two points? On what line do their centers lie?

5. What is meant by the statement, *Two points determine a line?* What points determine the perpendicular bisector of a line?

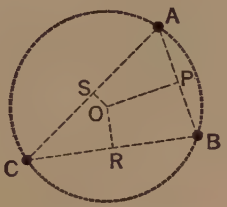


CIRCLES IN MACHINES

Circles play an important role in the construction of machines. Circular parts such as gears, bearings, pulleys, and wheels are used to transfer motion, to change the direction of motion, to increase the power or speed of moving parts, and to reduce friction.

Proposition 8 (C)

217. Theorem. *Three points not in a straight line determine a circle.*



Given: A, B, and C are three points not in a straight line.

To prove: A, B, and C determine a circle.

HINT: What theorem relates to the perpendicular bisectors of the sides of a triangle?

Proof: STATEMENTS	FACTS
1. Connect points A, B, and C to form $\triangle ABC$.	1. *Prin.* 13.
2. Draw perpendiculars to the sides of the triangle at the mid-points P, R, and S.	2. *Prin.* 15.
3. These lines meet at O, and O is equidistant from A, B, and C.	3. § 176.
4. Point O may be taken as the center of a circle with OA, OB, or OC as a radius.	4. *Prin.* 32.

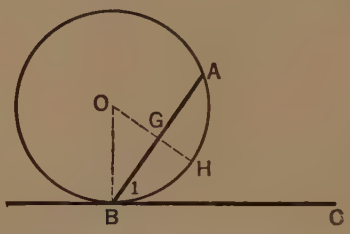
218. Corollary (c). *Two circles, or a straight line and a circle, can intersect in not more than two points.*

HINT: There cannot be three points of intersection because three points not in a straight line determine one circle.

CIRCLES AND THEIR ANGLES

Proposition 9 (c n)

219. Theorem. *An angle formed by a tangent and a chord drawn from the point of tangency equals numerically one-half its intercepted arc.*



Given: BC is a tangent and BA a chord of circle O.

To prove: $\angle 1 =$ numerically $\frac{1}{2}\overset{\frown}{AB}$.

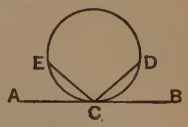
HINT: Show that $\angle 1$ equals a central angle whose arc is one-half $\overset{\frown}{AB}$. A central angle is equal numerically to what quantity?

Proof: STATEMENTS	FACTS
1. Draw radius OB.	1. By what Principle?
2. $BC \perp OB$.	2. Why?
3. Draw $OGH \perp BA$.	3. By what Principle?
4. $\angle 1 = \angle O$.	4. § 135.
5. $\angle O =$ numerically $\overset{\frown}{BH}$.	5. Why?
6. $\overset{\frown}{BH} = \overset{\frown}{HA} = \frac{1}{2}\overset{\frown}{BA}$.	6. Why?
7. $\angle 1 = \angle O =$ numerically $\overset{\frown}{BH} =$ numerically $\frac{1}{2}\overset{\frown}{BA}$.	7. Sub.

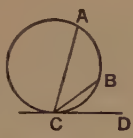
220. Corollary. *In the same circle (or in equal circles), if equal angles are formed by tangents and chords drawn from the points of tangency, their arcs are equal; and conversely.*

Exercises. Angles Formed by Tangents and Chords

1. Chords CD and CE make equal angles with tangent ACB. Prove $CD = CE$.

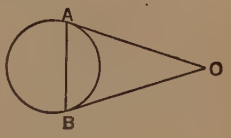


2. *Conversely,* if chords CD and CE are equal, they form equal angles with tangent ACB.

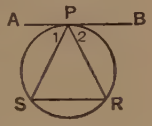


3. A line bisecting an angle formed by a tangent and a chord bisects the intercepted arc.

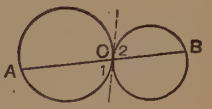
4. A chord forms equal angles with the tangents drawn at its extremities.



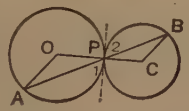
5. In the annexed figure, if $\triangle PRS$ is isosceles and APB is tangent to the circle, prove $\angle 1 = \angle 2$.



6. If two circles are tangent externally at O and AOB is drawn, arc OA equals numerically arc OB.

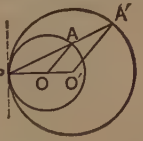


7. In the annexed figure, if circle O and circle C are tangent externally at P, prove $\angle O = \angle C$.

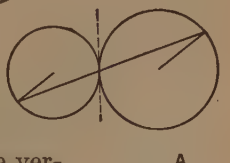


8. In the annexed figure, if the circles are tangent internally and APB is the common tangent, prove that $\overset{\frown}{PC}$ = numerically $\overset{\frown}{PD}$.

9. In the annexed figure, two circles are tangent internally at P and PAA' is any chord. Prove that radii OA and $O'A'$ are parallel.

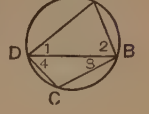


10. If two circles are tangent externally and a line is drawn through the point of contact and terminated by the circles, the radii drawn to its extremities are parallel.



221. Inscribed Angle. An angle whose vertex is on a circle and whose sides are chords is called an *inscribed angle.*

Name eight inscribed angles in the figure.



222. Segment of a Circle. The figure formed by an arc and its chord is called a *segment* of a circle. In the figure of § 221 there are ten segments of the circle. Name them.

223. An Angle Inscribed in a Segment. An angle is inscribed in a segment if its vertex lies on the arc of the segment and its sides pass through the extremities of that arc. $\angle C$, $\angle D$, and $\angle E$ are inscribed in the segment $ACDEB$.

Oral Exercises

1. How many segments of a circle are formed by drawing a chord?

2. How many angles may be inscribed in a segment?

3. May the angles of an inscribed polygon be called inscribed angles?

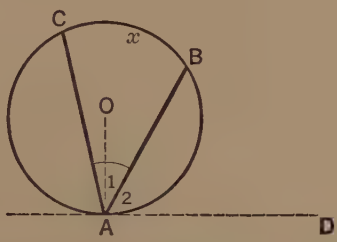
4. If an angle formed by a tangent and a chord is 80°, how many degrees are in its arc?

ROOSEVELT DAM *U. S. Reclamation Service*

Why is the breast of the Roosevelt Dam (Arizona) a great arc instead of a straight line? Is the curve concave or convex with respect to the water-pressure which comes from the right?

Proposition 10 (C N)

224. Theorem. *An inscribed angle equals numerically one-half its intercepted arc.*



Given: $\angle 1$ is an inscribed angle intercepting x of circle O.

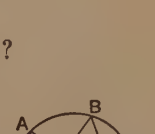
To prove: $\angle 1$ = numerically $\frac{1}{2}\,\widehat{x}$.

HINT: Use the Axiom of Subtraction.

Proof:

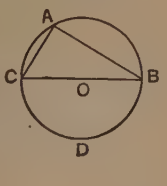
STATEMENTS	FACTS
1. Draw $AD \perp$ radius OA.	1. By what Principle?
2. AD is a tangent.	2. Why?
3. $\angle CAD$ = numerically $\frac{1}{2}\,\widehat{CBA}$ and $\angle 2$ = numerically $\frac{1}{2}\,\widehat{AB}$.	3. Why?
4. $\angle 1$ = numerically $\frac{1}{2}\,(\widehat{CBA} - \widehat{AB})$ = numerically $\frac{1}{2}\,\widehat{x}$.	4. Why?

225. Corollary 1 (c N). *Angles inscribed in a segment of a circle are equal.*



HINT: $\angle A$, $\angle B$, and $\angle C$ are each equal numerically to one-half what arc?

226. Corollary 2 (c n). *An angle inscribed in a semicircle is a right angle.*



HINT: If $\angle A$ is inscribed in the semicircle CAB, what kind of an arc is $\widehat{CDB}$? How is angle A measured?

227. Corollary 3. *In the same circle (or in equal circles), if inscribed angles are equal, their arcs are equal; conversely, if arcs are equal, their inscribed angles are equal.*

(The proof is left to the pupil.)

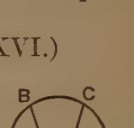
228. In §§ 219 and 224, we learn that angles formed by a tangent and a chord and that inscribed angles are equal numerically to *one-half their intercepted arcs.* From these facts and § 227 the following tool is derived:

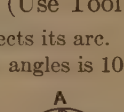
Tool XVI. Angles with vertices on a circle and with the same or equal arcs are equal; and conversely, arcs intercepted by equal angles whose vertices are on a circle are equal.

Exercises. Inscribed Angles. (Use Tool XVI.)

1. The bisector of an inscribed angle bisects its arc.

2. The difference between two inscribed angles is 10°. Find the difference between their respective central angles.

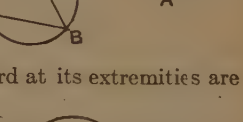




3. In the annexed figure, P is the midpoint of arc AB. Show that $\angle C$ is double $\angle PAB$

4. Two chords perpendicular to a third chord at its extremities are equal.

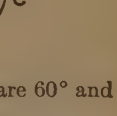
5. In the figure, prove the following pairs of triangles mutually equiangular: (*a*) $\triangle PBC$ and $\triangle PDA$; (*b*) $\triangle OAB$ and $\triangle OCD$.



6. The opposite angles of an inscribed quadrilateral are supplementary.

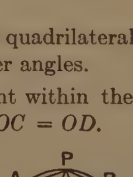
7. Two consecutive angles of an inscribed quadrilateral are 60° and 70°. Find the number of degrees in the other angles.

8. In the annexed figure, O is any point within the circle such that OA is equal to OB. Prove $OC = OD$.

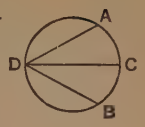


9. A parallelogram inscribed in a circle is a rectangle.

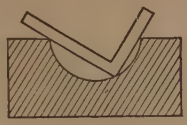
10. $ABCD$ is a square inscribed in a circle and P is any point on the arc AB. Prove that PC and PD trisect $\angle APB$.



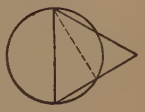
11. Arc AB is bisected by point C and CD is a diameter. Prove $AD = BD$.



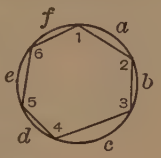
12. Pattern-makers use a square in testing a core-box to see whether it is a true semicircle. Upon what theorem is their method based?



13. A circle upon one of the equal legs of an isosceles triangle bisects the base. Prove.



14. Find the sum of the alternate angles of an inscribed hexagon.

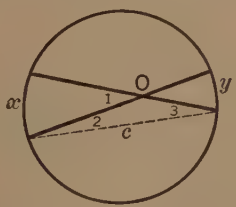


15. The angles of an inscribed triangle are 40°, 60°, and 80°. The bisectors of the angles meet the circle in X, Y, and Z. Find the angles of $\triangle XYZ$.

(Supplementary Exercises 31–45, p. 161, may be studied now.)

Proposition 11 (c n)

229. Theorem. *An angle formed by two chords intersecting within a circle equals numerically one half the sum of the arcs intercepted by it and its vertical angle.*



Given: $\angle 1$ and its vertical angle with $\overset{\frown}{x}$ and $\overset{\frown}{y}$ in the circle.

To prove: $\angle 1 =$ numerically $\frac{1}{2}(\overset{\frown}{x} + \overset{\frown}{y})$.

HINT: The exterior angle of a triangle equals the sum of what angles?

Proof: (The pupil may complete the demonstration.)

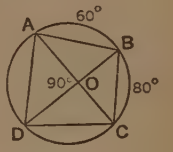
Exercises. Angles Formed by Intersecting Chords

1. From the annexed figure find the unknown values:

	$\angle 1$	$\overset{\frown}{x}$	$\overset{\frown}{y}$
a	?	60°	120°
b	?	72°	104°
c	65°	50°	?
d	78°	?	104°

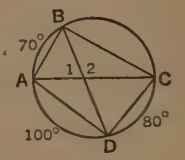
2. In the annexed figure, find the number of degrees in each angle of $ABCD$.

HINT: What special kinds of angles are in the figure?



3. If two chords intersecting within a circle are perpendicular, the sum of the opposite arcs equals a semicircle. Prove.

4. State the converse of Ex. 3. If it is true, prove it.



5. In the annexed figure, find $\angle A$, $\angle B$, $\angle C$, $\angle 1$, and $\angle 2$. What is the diagonal AC?

6. In the same figure, find $\angle CDB + \angle BCA + \angle 2$.

7. The vertices of an inscribed quadrilateral divide a circle into four arcs which have the ratio 3:4:5:6. Find (*a*) the number of degrees in each angle of the quadrilateral; (*b*) the number of degrees in the angles formed by the diagonals.

8. If $\overset{\frown}{CD} = \overset{\frown}{AD}$ and $\overset{\frown}{AH} = \overset{\frown}{BH}$, prove $AE = AG$.

HINT: Prove $\angle 1 = \angle 2$.

9. In the annexed figure, O is the center of the circle. Prove that the sum of angles AOC and BOD is double angle APC.

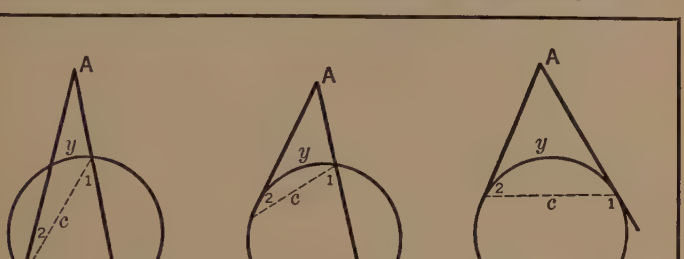
HINT: By what are the respective angles measured?

10. In the figure of Ex. 9, if $AO \perp PC$, $PB \perp DO$, $\angle CBP = 2\angle DOA$, prove $\overset{\frown}{AD} = 60°$ by using Principle 31; by using § 229.

11. Through a given point within a given circle draw three intersecting chords. Assign values to all but one of the intercepted arcs. Find the numerical values of the angles in your figure.

Proposition 12 (c n)

230. Theorem. *An angle formed by two secants, two tangents, or a tangent and a secant intersecting without a circle equals numerically one-half the difference of the intercepted arcs.*



Given: $\angle A$ intercepts arcs $\overset{\frown}{x}$ and $\overset{\frown}{y}$ in each figure.

To prove: $\angle A$ = numerically $\frac{1}{2}(\overset{\frown}{x} - \overset{\frown}{y})$.

HINT: The exterior angle of a triangle minus a non-adjacent interior angle equals what angle?

Proof: STATEMENTS	FACTS
1. Draw chord c.	1. By what Principle?
2. $\angle 1 = \angle 2 + \angle A$.	2. § 110.
3. $\angle A = \angle 1 - \angle 2$.	3. Axiom of Subtraction.
4. $\angle 1$ = numerically $\frac{1}{2}\overset{\frown}{x}$. $\angle 2$ = numerically $\frac{1}{2}\overset{\frown}{y}$.	4. § 224 or § 219.
5. $\angle A = \angle 1 - \angle 2$ = numerically $\frac{1}{2}(\overset{\frown}{x} - \overset{\frown}{y})$.	5. Sub.

Exercises. Angles Formed by Secants and Tangents

1. If two tangents are drawn from a point to a circle, find the intercepted arcs when the angle between the tangents is (*a*) 40°; (*b*) 60°; (*c*) 100°.

HINT for (*a*): $\overset{\frown}{x} + \overset{\frown}{y} = 360°$ and $\overset{\frown}{x} - \overset{\frown}{y} = 80°$.

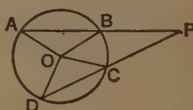
2. Find the arcs intercepted by two perpendicular tangents.

3. $\angle O$ is formed by two secants which intercept $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$. Find the parts not given in

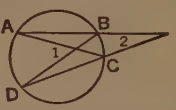
	$\angle O$	$\overset{\frown}{AB}$	$\overset{\frown}{CD}$
a	?	70°	110°
b	?	75°	125°
c	70°	90°	?
d	12°	?	32°

4. Find the angle formed by two tangents through the extremities of an arc of 50°; 60°; 40°; x°.

5. In the annexed figure, O is the center of the circle. Prove that $\angle AOD - \angle BOC = 2\ \angle P$.

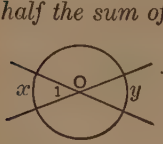


6. In the annexed figure, $\overset{\frown}{AD} = 60°$. Show that $\angle 1 + \angle 2 = 60°$.

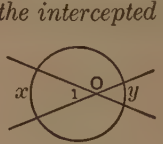


7. Find the angle formed by two tangents if the ratio of the intercepted arcs is 2 : 3.

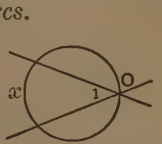
231. A Family of Theorems. The series of theorems from § 219 to § 230 together with Principle 31 constitute a related group or *family of theorems*. These theorems may be generalized in the following statement: *An angle formed by two intersecting lines which meet a circle equals numerically one-half the sum of the intercepted arcs.*



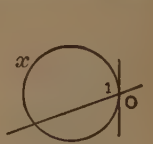
(*Prin* 31.)



(§ 229.)

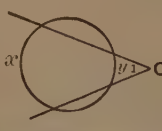


(§ 224.)



(§ 219.)

(§ 230.)



(These figures illustrate the generalized theorem. In the last three figures, O is outside the circle and $\widehat{y}$ has become negative, because it has decreased from positive values and passed through zero.)

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Patent Office
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Union Station
Columbus Memorial
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Capitol
Library of Congress
House Office Building
Supreme Court
Courthouse
Ellipse
Munitions Building
Navy Building
Washington Monument
Lincoln Memorial
National Museum
Smithsonian Institution
Bureau of Engraving & Printing
Botanic Garden
Grant Memorial
Agriculture
Fisheries
Army Medical Museum
National Gallery of Art
Tidal Basin
Potomac River
Potomac Park
Polo Grounds
Garfield Park
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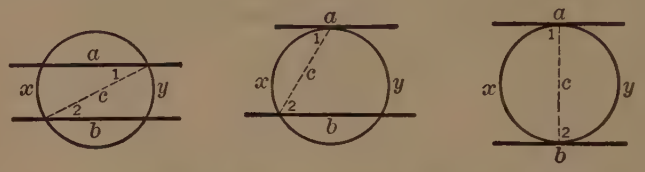
Proposition 13 (c n)

232. Theorem. *Arcs of a circle included between parallel lines are equal.*



Given: $a \parallel b$, $\overset{\frown}{x}$ and $\overset{\frown}{y}$ are included arcs in each figure.
To prove: $\overset{\frown}{x} = \overset{\frown}{y}$.

HINT: Draw a chord to form angles whose arcs are x and y. How are the angles related?

Proof:

STATEMENTS	FACTS
1. Draw chord c.	1. By what Principle?
2. $a \parallel b$.	2. Why?
3. $\angle 1 = \angle 2$.	3. Why?
4. $\angle 1$ = numerically $\frac{1}{2}\overset{\frown}{x}$ and $\angle 2$ = numerically $\frac{1}{2}\overset{\frown}{y}$.	4. Why?
5. $\overset{\frown}{x} = \overset{\frown}{y}$.	5. Sub. and *Prin.* 7.

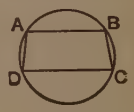
Exercises. Parallel Lines and Their Arcs

1. A trapezoid inscribed in a circle is isosceles.

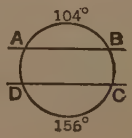
2. Lines which intercept equal arcs of a circle are parallel (provided they do not intersect within the circle).

3. If two equal chords intersect within a circle, the lines joining their ends form an isosceles trapezoid.

4. If $AD = BC$, what kind of quadrilateral is $ABCD$? Prove your answer.

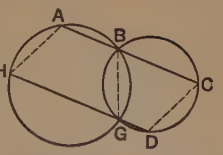


5. In the annexed figure, $AB \parallel CD$. Find arcs AD and BC.



6. In the figure of Ex. 5, find the angle formed by producing DA and CB until they meet.

7. Two parallel lines drawn through the points of intersection of two circles and limited by the circles are equal.



HINT: Prove AH and CD both equal and parallel.

8. The line which joins the mid-points of two parallel chords is perpendicular to each of the chords.

SUPPLEMENTARY EXERCISES

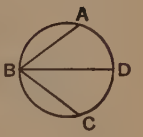
Central Angles, Arcs, and Chords

1. In the annexed figure, if $AD = BC$ show that $BD = AC$.

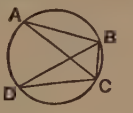
2. State the converse of Ex. 1 and prove it.

3. AB, BC, and CD are equal chords and OA, OB, OC, and OD are radii of the circle. Prove $\angle AOC = \angle BOD$.

4. If the diameter BD bisects the arc AC, prove $AB = BC$.



5. On a circle four points, A, B, C, and D are so placed that AB and CD are equal chords. Prove that chords AC and BD are equal.



6. If $AB = CD$, prove $\triangle ABC \cong \triangle BDC$.

7. The diagonals of an equilateral pentagon inscribed in a circle are equal.

8. In the annexed figure, if the perpendiculars to two radii from a point on the circle are equal, the point bisects the arc between the radii.

9. O is the center of the circle shown, with G and H the mid points of the radii and $CG = CH$. Prove $\overset{\frown}{AC} = \overset{\frown}{BC}$.

10. The radii drawn to the vertices of an inscribed equilateral hexagon divide the figure into six equilateral triangles.

11. Equal parallel chords intercept two pairs of equal arcs.

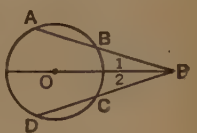
12. If two equal circles intersect, the arcs cut off on each circle by the other are equal.

HINT: Draw radii and the common chord.

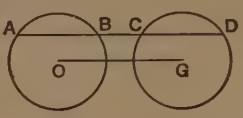
Chords and Their Distances from Centers

13. Two parallel chords drawn from the extremities of a diameter are equal.

14. In the annexed figure, O, the center, lies on the bisector of angle P. Prove $\overset{\frown}{AB} = \overset{\frown}{CD}$.



15. The chords cut off by two equal circles from a straight line parallel to the line joining their centers are equal.



HINT: Draw perpendiculars from O and G to AD.

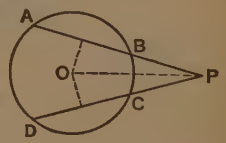
16. The chords cut off by two equal circles from a straight line bisecting the line joining their centers are equal.

17. If a secant intersects two concentric circles, the segments limited by the two circles are equal.

HINT: Draw $OP \perp AD$.

18. If two equal chords intersect each other, the line joining their point of intersection with the center bisects an angle formed by the chords.

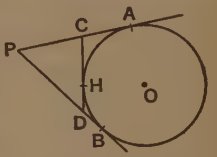
19. If two equal chords of a circle are produced to meet at P, the secants from P are equal.



Tangents

20. If an isosceles triangle is inscribed in a circle, the tangents at the vertices form another isosceles triangle.

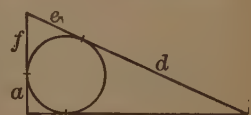
21. If PA and PB are tangents from P, and CD is tangent at any point of arc AB, the perimeter of $\triangle PCD$ equals the sum of the tangents PA and PB.



22. Using the figure of Ex. 21, prove $\angle P$ and $\angle AOB$ supplementary.

23. Using the figure of Ex. 21, prove $\frac{1}{2}\angle P$ and $\angle COD$ complementary.

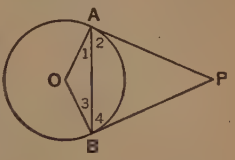
24. The diameter of the circle inscribed in a right triangle is equal to the difference between the sum of the legs and the hypotenuse.



HINT: Prove $a + b = 2r$.

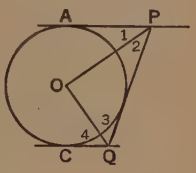
25. Two tangents form an angle of 120° if the distance from the center of the circle to the point of intersection of the tangents is equal to the sum of the tangents.

26. In the annexed figure, PA and PB are tangents to a circle whose center is O. Prove that $\angle P$ is double $\angle OAB$.



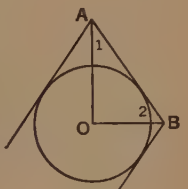
HINT: Show $\angle 1 + \angle 2 + \angle 3 + \angle 4 = \angle 2 + \angle 4 + \angle P$.

27. In the figure of Ex. 26, prove that OP bisects arc AB.



28. In the figure of Ex. 26, prove that the perpendicular bisector of AB passes through O and P.

29. In the annexed figure, AP and CQ are parallel tangents meeting a third tangent at P and Q. If O is the center, show that $\angle POQ$ is a right angle.



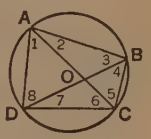
HINT: $\angle 1 + \angle 2 + \angle 3 + \angle 4 = 180°$, $\angle 1 = \angle 2$, etc.

30. If two perpendicular radii of a circle when produced cut a tangent at points A and B, the other tangents from A and B to the circle are parallel.

HINT: Show $\angle A + \angle B = 180°$.

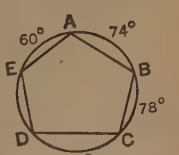
Inscribed Angles

31. Select from the figure the pairs of numbered angles which are equal. What is the reason for each equality?



32. In the figure of Ex. 31, prove a pair of triangles mutually equiangular.

33. In the annexed figure, find each angle of the inscribed pentagon. What check may be applied to the sum of the angles?

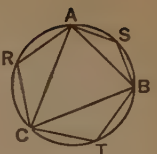


34. An equilateral quadrilateral inscribed in a circle is a square.

35. If the arcs of the sides of an inscribed triangle have the ratio 2:4:6, what special triangle is represented?

HINT: $2x + 4x + 6x = 360°$.

36. Find the angles of an inscribed triangle if the arcs subtended by the sides have the ratio 5:7:12.



37. In the annexed figure, prove $\angle R + \angle S + \angle T = 360°$.

HINT: $2\ \angle R = \overset{\frown}{AB} + \overset{\frown}{BC}$.

38. Two triangles are congruent if they are inscribed in equal circles and have their angles respectively equal.

39. If equal chords intersect, their segments are respectively equal.

HINT: Draw AD and BC. Prove $\triangle OAD \cong \triangle OBC$.

40. A chord is parallel to a tangent drawn through the mid-point of the arc of the chord.

a) Prove by Fig. 1.

b) Prove by Fig. 2.

Fig. 1

Fig. 2

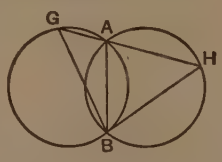
41. In the annexed figure, O is the center with OP perpendicular to chord AB. Prove $\angle O$ equal to $\angle C$.

42. If from any point on a circle a chord and a tangent are drawn, the perpendiculars drawn to them from the mid-point of the included arc are equal.

43. If the bisector of an inscribed angle intersects the circle and a chord is drawn through the point of intersection parallel to one side of the angle, it is equal to the other side of the angle.

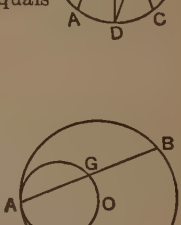
HINT: Prove that DE which is parallel to AB equals BC. Find the relation of arcs AD, BE, and DC.

44. Two equal circles intersect each other at A and B. Through A a line is drawn terminated by the circles at G and H. Prove $BG = BH$.



HINT: Prove $\angle G = \angle H$.

45. In the annexed figure, two circles are tangent at A and the smaller circle passes through O, the center



of the larger. Show that any chord of the larger circle from A, such as AB, is bisected by the smaller circle.

HINT: Draw OG.

SUMMARY OF CHAPTER TWO

In the same circle (or in equal circles) **lines are equal,** if they are

(1) radii or diameters;
(2) chords with equal central angles;
(3) chords with equal arcs;
(4) chords equidistant from the center;
(5) perpendiculars from the center to equal chords;
(6) segments of a chord formed by a radius perpendicular to it.
(7) tangents from the same external point.

In the same circle (or in equal circles) **angles are equal,** if they are

(1) central angles with equal arcs;
(2) central angles with equal chords;
(3) angles with their vertices on a circle and with the same or equal arcs;
(4) angles with equal numerical measures;
(5) angles inscribed in the same segment.

In the same circle (or in equal circles) **arcs are equal,** if they are

(1) arcs of equal central angles;
(2) arcs of equal chords;
(3) arcs formed by a radius perpendicular to a chord;
(4) arcs cut off between two parallels;
(5) arcs with equal angles whose vertices are on the circle.

Angles Measured by Arcs

1. Central angles have the same ratio as their arcs.

2. A central angle equals numerically its intercepted arc.

3. An inscribed angle equals numerically one-half its intercepted arc.

4. An angle formed by two chords intersecting within a circle equals numerically one-half the sum of the arcs intercepted by it and its vertical angle.

5. An angle formed by a tangent and a chord equals numerically one-half the intercepted arc.

6. An angle formed by two secants, two tangents, or a tangent and a secant intersecting at an external point, equals numerically one-half the difference of the intercepted arcs.

REVIEW ORAL EXERCISES. DIAGNOSTIC TEST.

1. What is the difference between (*a*) a radius and a diameter? (*b*) a chord and a secant? (*c*) a secant and a tangent?

2. If a diameter of a circle is 10 in., what is the length of (*a*) a radius? (*b*) the longest chord?

3. How may a chord be changed to a secant?

4. If a line moves across a circle in such a way that it is always parallel to its original position, in how many positions may it be (*a*) a tangent? (*b*) a secant?

5. (*a*) How many chords may be drawn from a point on a circle? (*b*) how many diameters?

6. (*a*) How many tangents may be drawn to a circle from an external point? (*b*) how many secants?

7. (*a*) May a straight line have one point in common with a circle? two points? three points? (*b*) What is the line called in each case?

8. A circle may have how many diameters through any point on it? how many tangents? how many secants? how many chords?

9. What lines of a circle are necessarily equal? Which lines may be unequal?

10. Are two lines necessarily equidistant from the center if they are (*a*) chords? (*b*) tangents? (*c*) secants?

11. Are two circles equal if they have (*a*) equal radii? (*b*) equal chords? (*c*) equal diameters?

12. How many points are required to determine (*a*) a straight line? (*b*) two parallel lines? (*c*) an angle? (*d*) a circle?

13. How many circles may pass through one point? two points? three points?

14. Why are two tangents to a circle equidistant from the center?

15. How many circles can pass through three points in a straight line? through three points not in a straight line?

16. (*a*) In what two ways may chords be proved equal? (*b*) In what three ways may arcs be proved equal?

17. Can arcs of unequal circles be equal in degrees? Can they be made to coincide?

18. If two chords bisect each other, what kind are they?

19. (*a*) If two diameters intersect, into how many arcs is the circle divided? (*b*) Why are the opposite arcs equal? (*c*) What is true of any two consecutive arcs?

20. (*a*) If a line bisects a chord will it bisect an arc of the chord? (*b*) If a line is perpendicular to a chord will it bisect an arc of the chord?

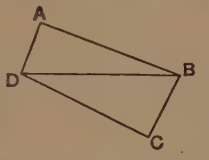
21. How many tangents may be drawn to a circle through a point within a circle? through a point on the circle? through a point without the circle?

22. May two circles have one point in common? two points? three points? In each case describe the relation of the circles.

MISCELLANEOUS EXERCISES

1. A line which bisects the two arcs of a chord is a diameter.

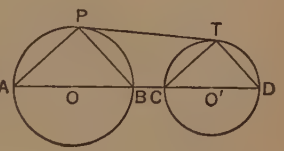
2. A circle with the hypotenuse of a right triangle as its diameter passes through the vertex of the right angle.



3. If A and C of quadrilateral $ABCD$ are right angles, diagonal BD is the diameter of a circle which passes through A and C.

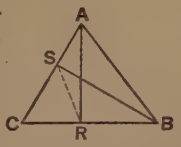
4. If RA and SB are perpendicular to chord AB at its extremities, they cut off equal segments CR and SD on diameter CD.

HINT: Show $\triangle TOR \cong \triangle BOS$.



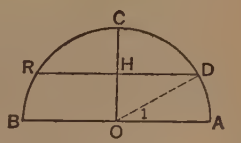
5. In the annexed figure, O and O' are the centers of the circles and PT is an external tangent. Prove that AP and PB are parallel to CT and DT.

6. AR and BS are altitudes of triangle ABC. Prove $\angle ABS = \angle ARS$.



HINT: R and S lie on a circle whose diameter is AB.

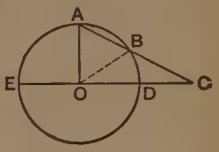
7. O is the center of a semicircle, CO is perpendicular to AB, RD is the perpendicular bisector of CO. Prove $\overset{\frown}{CD} = 2\,\overset{\frown}{DA}$.



HINT: $OH = \frac{1}{2}\,OD$, $\angle 1 = 30°$.

8. If a square is drawn upon the hypotenuse of a right triangle, a line from the center of the square to the vertex of the right angle bisects the right angle.

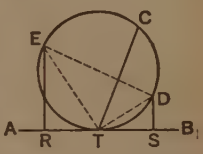
9. If two circles are tangent externally and two lines are drawn through the point of contact and are terminated by the circles, the chords joining the ends of these lines are parallel.



10. In the annexed figure, $BC = OD$ and O is the center of the circle. Prove $\angle AOE = 3\angle C$.

11. If two circles are tangent to each other, the line of centers passes through the point of contact.

12. AB is tangent to a circle at T and CT is any chord through T. D and E are mid-points of the arcs formed by CT. DS and ER are perpendicular to the tangent. Prove $DS + ER =$ the diameter of the circle.



13. In the annexed figure, $AS \perp CB$ and $BR \perp AC$. Prove $\overset{\frown}{RC} = \overset{\frown}{CS}$.

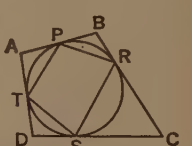
HINT: $\angle 1$ and $\angle 2$ are each numerically equal to what arcs?

14. If a chord is parallel to a diameter, the intercepted arcs are equal. (Prove independently of Prop. 13.)

15. AB and CD are two chords of a circle. M, the mid-point of $\overset{\frown}{AB}$, is connected with C and D. Prove $\triangle GHM$ and $\triangle CDM$ mutually equiangular.

16. In the figure of Ex. 15, prove that the opposite angles of quadrilateral $CDGH$ are supplementary.

17. In the annexed figure, if $\overset{\frown}{PR} = 64°$, $\overset{\frown}{RS} = 126°$, and $\overset{\frown}{ST} = 78°$, find the three angles at P.



18. With the same figure and conditions as Ex. 17, find the four angles of the circumscribed quadrilateral.

CHAPTER THREE

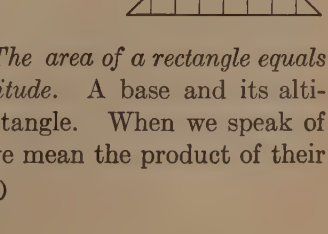
AREAS AND PROPORTIONS

AREAS

233. Units of Measure. The pupil is familiar with the fact that the area of a figure bounded by a closed line is expressed in square units. The square units generally correspond with the linear units of measure which are used. For example, we measure the length and the width of a sheet of paper in inches and give the area in *square inches.* We measure a room in feet and give its area in *square feet.* The boundaries of a state are measured in miles and its area is expressed in *square miles.*

234. Unit of Surface. Area of a Surface. A *unit of surface* is a square whose side is a *unit of length.* For example, a square inch is a *one-inch square* and a square foot is a *one-foot square.* The *area of a surface* is the number of *units of surface* it contains. When we say that the area of a surface is 16 square inches we mean that it contains 16 *one-inch squares* or their equivalent.

In the square and the rectangle shown, there are 16 complete *one-inch squares;* the triangle contains only the *equivalent* of 16 *one-inch squares.*



235. Principle 39 states that, *The area of a rectangle equals the product of a base and its altitude.* A base and its altitude are the *dimensions* of a rectangle. When we speak of the product of two dimensions we mean the product of their

numerical measures. If the base and the altitude of the rectangle in § 234 in terms of inches are 8 and 2, the area expressed in square inches is 8×2 or 16. This fact is of such importance in geometry that it is now given as

Tool XVII. Area of a Rectangle = Base $\times$ Altitude.

236. Equal Figures. Figures are said to be *equal* or *equivalent* if their *areas* are equal. The square, the rectangle, and the triangle shown in § 234 are *equal* or *equivalent.*

Oral Exercises

1. Are congruent figures always equal? What does the symbol of congruence indicate?

2. Are equal figures always congruent? Illustrate.

3. If a rectangle is equal to a square containing 36 sq. in., what are some of the possible dimensions of the rectangle?

4. What is the area of a rectangle (*a*) 3 ft. by 24 in.; (*b*) 4 yd. by 72 in.; (*c*) 6 ft. by 6 yd.?

5. In the annexed figure, find the approximate areas of the parallelogram and the triangle by counting the squares. Count any square as one if half or more than half of it is within the figure.

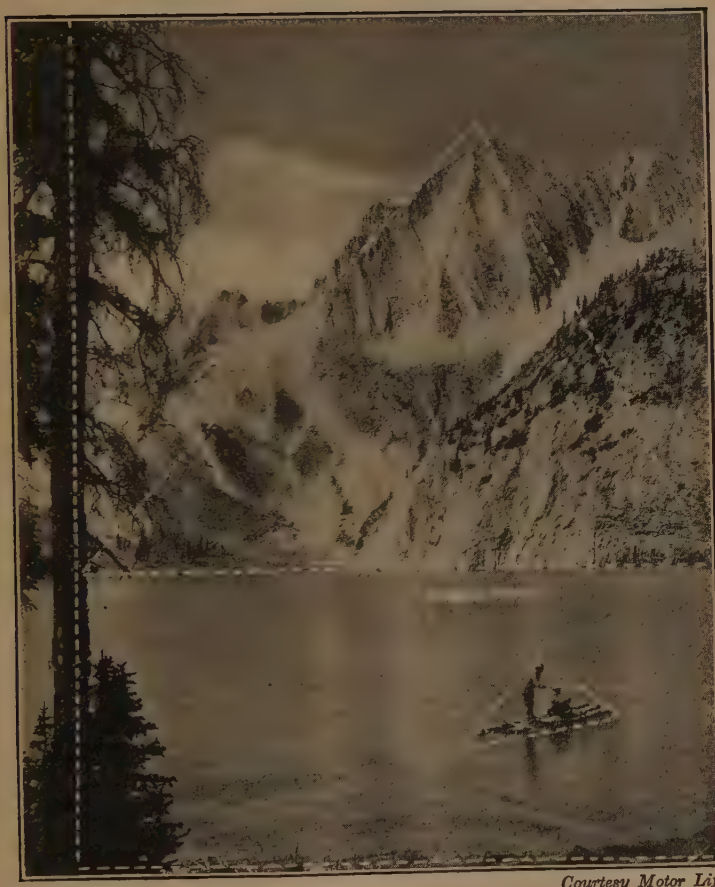
6. Find the approximate area of each of the following figures:

7. If $AB = 2$ in. and $CD = 3$ in., what is the value of the ratio of AB to CD?

8. What is the value of the ratio of 4×7 to 2×21? (Answer, $\frac{2}{3}$.) Why?

9. Write as a fraction the ratio of the product of a and b to the product of a' and b'.

10. If $R = a \times b$ and $S = a' \times b'$, what is the ratio of R to S in terms of these products?



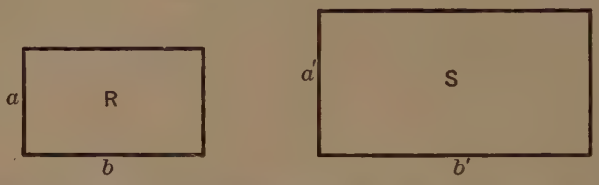
Courtesy Motor Life

GEOMETRY IN PICTURES

The composition of a good landscape depends upon its lines and its mass. These, according to a good authority, are made up of seven elementary forms: the balance, triangle, circle, square, perpendicular lines, curving line, and radii. In the picture above observe that the perpendicular lines suggest stability; the balance between two objects, the perspective; and the triangle, dignity and repose.

Proposition 1

237. Theorem. *Rectangles have the same ratio as the products of their bases by their altitudes.*



Given: R and S are rectangles with altitudes a and a' and with bases b and b'.

To prove: $\frac{R}{S} = \frac{ab}{a'b'}$.

HINT: What Principle relates to a rectangle? What operation is indicated by $\frac{R}{S}$?

Proof:

STATEMENTS	FACTS
1. $R = ab$ and $S = a'b'$.	1. Why?
2. $\frac{R}{S} = \frac{ab}{a'b'}$.	2. Axiom of Division.

238. Corollary 1. *Rectangles with equal altitudes have the same ratio as their bases.*

239. Corollary 2. *Rectangles with equal bases have the same ratio as their altitudes.*

240. Corollary 3. *Rectangles with equal bases and equal altitudes are equal.*

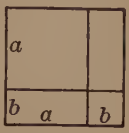
HINT: Show that the ratio of the rectangles is equal to unity and apply *Prin.* 2.

Exercises. Areas of Rectangles

1. How many tiles are required to cover a floor 16 ft. by 24 ft. if the tiles are (*a*) 6 in. square? (*b*) 8 in. square?

2. Find the dimensions of a rectangle whose perimeter is 26 and whose area is 36.

HINT: $x + y = 13$, $xy = 36$.



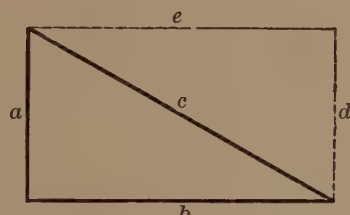
3. Show geometrically that $(a + b)^2 = a^2 + 2\,ab + b^2$.

4. The length of a rectangle is twice the width. Find the area when the perimeter is (*a*) 12; (*b*) 18; (*c*) 20; (*d*) p.

(Supplementary Exercises 1–9, p. 198, may be studied now.)

Proposition 2

241. Theorem. *The area of a right triangle equals one-half the product of its legs.*



Given: rt. $\triangle abc$ with legs a and b.

To prove: $\triangle abc = \frac{1}{2}\, a \times b$.

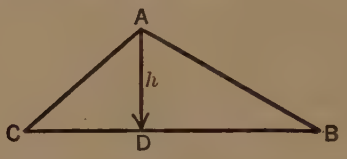
HINT: Form a rectangle. How is the triangle related to the rectangle?

Proof:

STATEMENTS	FACTS
1. Draw $e \parallel b$ and $d \parallel a$.	1. *Prin.* 14.
2. $abde$ is a rectangle.	2. Def.
3. $\triangle abc = \triangle cde = \frac{1}{2}\, abde$.	3. § 146.
4. Rectangle $abde = a \times b$.	4. *Prin.* 39.
5. $\triangle abc = \frac{1}{2}\, abde = \frac{1}{2}\, a \times b$.	5. Sub.

Proposition 3 (c N)

242. Theorem. *The area of a triangle equals one-half the product of a base and its altitude.*



Given: h is the altitude and BC the base of $\triangle ABC$.

To prove: $\triangle ABC = \frac{1}{2} h \times BC$.

HINT: What kind of triangles are formed by an altitude of a triangle?

Proof: STATEMENTS	FACTS
1. $h \perp CB$.	1. Given.
2. $\triangle ADC$ and $\triangle ADB$ are right triangles.	2. Def.
3. Rt. $\triangle ADC = \frac{1}{2} h \times CD$, and rt. $\triangle ADB = \frac{1}{2} h \times DB$.	3. Why?
4. $\triangle ABC = \frac{1}{2} h(CD + DB)$.	4. Axiom of Addition.
5. $CD + DB = BC$.	5. Why?
6. $\triangle ABC = \frac{1}{2} h(CD + DB) = \frac{1}{2} h \times BC$.	6. Sub.

243. Corollary 1. *Triangles have the same ratio as the products of their bases and altitudes.*

244. Corollary 2. *Triangles with equal* $\left\{ \begin{matrix} \text{altitudes} \\ \text{bases} \end{matrix} \right\}$ *have the same ratio as their* $\left\{ \begin{matrix} \text{bases} \\ \text{altitudes} \end{matrix} \right\}$.

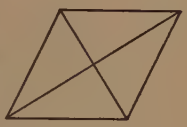
245. Corollary 3 (c n). *Triangles with equal bases and equal altitudes are equal.*

246. Corollary 4. *Triangles with a common base and with their vertices in a line parallel to the base are equal.*

247. Corollary 5. *The line joining the vertices of equal triangles (on the same side of a common base) is parallel to the base.*

248. Corollary 6. *The area of a rhombus equals one-half the product of its diagonals.*

HINT: The diagonals are perpendicular bisectors of each other.



249. Corollary 7. *The area of a square equals one-half the square of its diagonal.*

250. Corollary 8. *The area of a square equals the square of a side.*

251. Corollary 9. *If s is the side of a square and d its diagonal, $d = s\sqrt{2}$ and $s = \frac{1}{2}d\sqrt{2}$.*

HINT: $ABCD = s^2$. Why?

$ABCD = \frac{d^2}{2}$. Why? (To be completed by the pupil.)

252. Tool XVIII. Area of a triangle $= \frac{\text{Base} \times \text{Altitude}}{2}$.

Exercises. Areas of Triangles

1. Find the area of a triangle if its base and its altitude are respectively 9 and 6; 10 and 7; 12 and 16.

2. The base and the altitude of one triangle are 24 and 16, of another triangle, 36 and 20. Find the ratio of their areas.

3. The area of a triangle is 54 sq. in. Find its altitude if its base is (*a*) 9 in.; (*b*) 12 in.; (*c*) 18 in.; (*d*) $13\frac{1}{2}$ in.

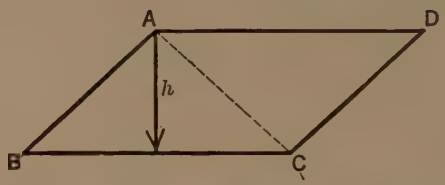
HINT: Area $= \frac{\text{Base} \times \text{Altitude}}{2}$, or $54 = \frac{9h}{2}$. Solve for h.

4. In any right triangle the product of the legs equals the product of the hypotenuse and the altitude upon the hypotenuse.

(Supplementary Exercises 10–18, p.199, may be studied now.)

Proposition 4 (C N)

253. Theorem. *The area of a parallelogram equals the product of its base and altitude.*



Given: h is the altitude and BC the base of $\square\ ABCD$.

To prove: $ABCD = h \times BC$.

HINT: Draw a diagonal. How is the parallelogram related to each of the triangles formed?

Proof: STATEMENTS	FACTS
1. Draw AC.	1. *Prin.* 13.
2. $\triangle ABC = \triangle ACD = \frac{1}{2} \square ABCD$.	2. § 146.
3. $\triangle ABC = \frac{1}{2} h \times BC$.	3. Why?
4. $\triangle ABC = \frac{1}{2} \square ABCD = \frac{1}{2} h \times BC$.	4. Sub.
5. $ABCD = h \times BC$.	5. Axiom of Multiplication.

254. Corollary 1. *Parallelograms have the same ratio as the products of their bases by their altitudes.*

255. Corollary 2. *Parallelograms with equal* $\left\{ \begin{matrix} \textit{altitudes} \\ \textit{bases} \end{matrix} \right\}$ *have the same ratio as their* $\left\{ \begin{matrix} \textit{bases} \\ \textit{altitudes} \end{matrix} \right\}$.

256. Corollary 3 (c n). *Parallelograms with equal bases and equal altitudes are equal.*

Exercises. Areas of Parallelograms

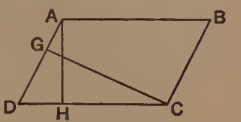
1. A rectangle and a parallelogram are equal. If their bases are equal, how do their altitudes compare?

2. If one parallelogram has half the base and twice the altitude of another parallelogram, how do their areas compare?

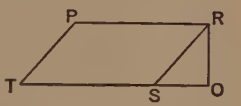
HINT: For the bases use b and $2b$, and $2a$ and a for the altitudes.

3. If a parallelogram and a triangle have equal areas and equal bases, how do their altitudes compare?

4. In the annexed figure, $ABCD$ is a parallelogram with perpendiculars AH and CG. Prove $AH \times CD = CG \times AD$.

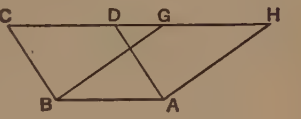


5. In the figure, $PRST$ is a $\square$ and $OR \perp OS$. If $OS = 4$, $ST = 10$, and $RO = 6$, find the ratio of the area of the triangle to the area of the parallelogram.



6. Prove by a theorem on areas that a parallelogram is divided into two equal triangles by a diagonal.

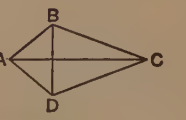
7. In the annexed figure, if $CH \parallel BA$, $CB \parallel DA$, and $GB \parallel HA$, prove $ABCD = ABGH$.



8. In the same figure, prove $\triangle BCG = \triangle ADH$.

9. In the parallelogram $ABCD$, E bisects DC. Show that ADE is one-fourth of the parallelogram.

10. The area of the kite-shaped figure $ABCD$ equals one-half the product of its diagonals.



11. $ABCD$ is a parallelogram whose diagonals intersect at O. Find three triangles equal to $\triangle OAB$; find three triangles equal to $\triangle ABC$.

12. $ABCD$ is a parallelogram with P a point in AB. Prove that $\triangle PCD = \frac{1}{2}\ ABCD$.

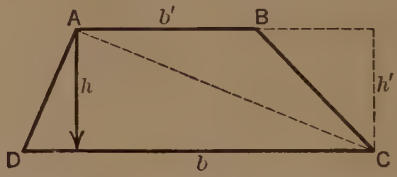
13. A side of a rhombus is 5; its diagonals are 6 and 8. Find an altitude.

14. The legs of a right triangle are 6 and 8 and the hypotenuse is 10. Find the third altitude.

(Supplementary Exercises 19–28, p. 203, may be studied now.)

Proposition 5 (C N)

257. Theorem. *The area of a trapezoid equals one-half the product of an altitude by the sum of the bases.*



Given: $ABCD$ is a trapezoid, h is an altitude, b and b' are bases.

To prove: $ABCD = \frac{1}{2}h(b + b')$.

HINT: Divide the figure into two triangles. May the bases of the trapezoid be taken as bases of the triangles?

Proof: STATEMENTS	FACTS
1. Draw diagonal AC.	1. *Prin.* 13.
2. $\triangle ACD = \frac{1}{2}h \times b$.	2. Why?
3. Draw $h' \perp AB$ produced.	3. *Prin.* 15.
4. $h' \parallel h$.	4. *Prin.* 17.
5. $h' = h$.	5. Why?
6. $\triangle CAB = \frac{1}{2}h'b'$ or $\frac{1}{2}hb'$.	6. Why?
7. $ABCD = \frac{1}{2}hb + \frac{1}{2}hb' = \frac{1}{2}h(b + b')$.	7. Axiom of Addition.

258. Corollary. *The area of a trapezoid equals the product of its altitude and its median.*

259. Correlation of Area Formulas.

The area of a rhombus or the area of a square equals one-half the product of its diagonals. $S = \frac{1}{2}dd'$.

The area of a parallelogram or the area of a rectangle equals the product of a base and its altitude. $S = hb$.

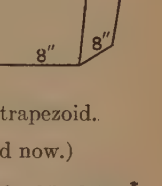
The area of a trapezoid or the area of a triangle equals one-half the product of an altitude by the sum of the bases. (In the triangle one of the bases is zero.) $S = \frac{1}{2}h(b + b')$.

Exercises. Areas of Trapezoids

1. In the figure, $BC \parallel DE$ and $AH \perp DE$. $BC = 15$, $DE = 20$, $AG = 9$, and $AH = 12$. Find the area of the trapezoid $CBDE$ by the regular formula and also by the difference of the areas of the two triangles.

2. A diagonal of a trapezoid divides the figure into two triangles whose ratio equals the ratio of the bases of the trapezoid.

3. A manual training department constructs 50 waste baskets as shown in the figure. How many square feet of wood are in the finished baskets, the sides being isosceles trapezoids 11″ high?



4. If the boards used in the preceding exercise were one foot wide and twelve feet long, what is the least number of boards required for the sides?

5. A line bisecting the bases of a trapezoid bisects the trapezoid.

(Supplementary Exercises 29–34, p. 201, may be studied now.)

260. Tool XIX. Triangles with the same or equal $\left\{\begin{matrix}\text{altitudes}\\\text{bases}\end{matrix}\right\}$ have the same ratio as their $\left\{\begin{matrix}\text{bases}\\\text{altitudes}\end{matrix}\right\}$.

There are two important applications of Tool XIX. By it we may show

(1) that the ratio of two triangles equals the ratio of two corresponding lines, as $\frac{\triangle 1}{\triangle 2} = \frac{a}{b}$;

(2) that the ratio of two lines equals the ratio of two other lines, as $\frac{a}{b} = \left(\frac{\triangle 1}{\triangle 2}\right) = \frac{c}{d}$.

In the first case, if the two triangles of the conclusion have the same or equal altitudes (or bases), make the ratio of the two triangles equal to the ratio of their corresponding bases (or altitudes).

The second case is a double application of Case I and the conclusion is brought about by substitution.

Exercises. Ratios of Lines of Triangles

1. By means of Tool XIX and the directions which accompany it find the following ratios in Fig. 1:

a) $\frac{\triangle 1}{\triangle ABC} = \frac{?}{?}$; *c)* $\frac{\triangle 1}{\triangle 2} = \frac{?}{?}$.

b) $\frac{\triangle 2}{\triangle ABC} = \frac{?}{?}$;

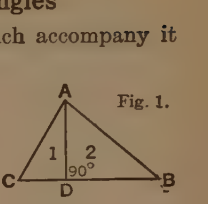


Fig. 1.

2. In like manner find the corresponding ratios in Fig. 2.

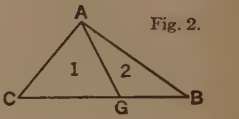


Fig. 2.

3. Find the corresponding ratios in Fig. 3.

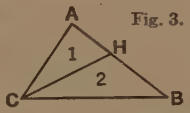


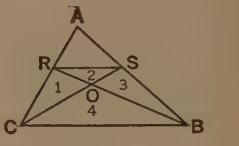
Fig. 3.

4. In the annexed figure, $RS \parallel BC$. Prove the following equalities of ratios:

a) $\frac{OS}{OC} = \frac{\triangle 3}{?}$; *d)* $\frac{\triangle CRS}{\triangle BRS} = 1$;

b) $\frac{OS}{OC} = \frac{\triangle 2}{?}$; *e)* $\frac{\triangle 2}{\triangle RSB} = \frac{?}{RB}$.

c) $\frac{\triangle 2}{\triangle 1} = \frac{?}{\triangle 4}$;

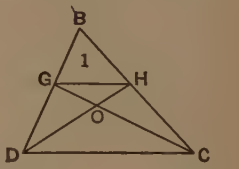


5. In the annexed figure, if $GH \parallel DC$, find a ratio equal to each of the following ratios:

a) $\triangle 1 : \triangle BCG$; *d)* $BH : CH$;

b) $\triangle BDH : \triangle 1$; *e)* $\triangle GDO : \triangle GOH$;

c) $BD : BG$; *f)* $\triangle HOC : \triangle GOH$.



6. From parts (*b*) and (*a*) of Ex. 5, find $\frac{\triangle BDH}{\triangle BCG} = \frac{? \times ?}{? \times ?}$.

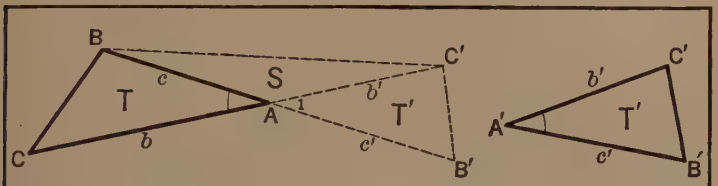
HINT: Find values for (*b*) and (*a*) and use the Axiom of Multiplication.

7. From parts (*e*) and (*f*) of Ex. 5, find $\frac{\triangle GDO}{\triangle HOC} = \frac{? \times ?}{? \times ?}$.

8. (*a*) Make a sketch of the figure in Ex. 4 above; (*b*) draw the common altitude of $\triangle 1$ and $\triangle 2$; (*c*) complete the equality, $\frac{\triangle 1}{\triangle 2} = \frac{?}{?}$; (*d*) draw the common altitude of $\triangle 2$ and $\triangle 3$; (*e*) complete the equality, $\frac{\triangle 2}{\triangle 3} = \frac{?}{?}$; (*f*) prove that $\frac{\triangle 1}{\triangle 3} = \frac{OC \times OR}{OB \times OS}$ by using the Axiom of Multiplication.

Proposition 6

261. Theorem. *Triangles with an angle of one equal to an angle of another have the same ratio as the products of the sides including the equal angles.*



Given: $\triangle T$ and T' with $\angle A = \angle A'$.

To prove: $\dfrac{\triangle T}{\triangle T'} = \dfrac{bc}{b'c'}$.

HINT: What is equal to the ratio of two triangles which have a common altitude? Have $\triangle T$ and $\triangle S$ a common altitude? Have $\triangle S$ and $\triangle T'$?

Proof: STATEMENTS	FACTS
1. Produce CA and BA through A forming $\angle 1$.	1. *Prin.* 11.
2. $\angle 1 = \angle A$.	2. Why?
3. $\angle A = \angle A'$.	3. Why?
4. $\angle 1 = \angle A = \angle A'$.	4. Why?
5. Place $\triangle T'$ so that $\angle A'$ coincides with $\angle 1$.	5. *Prin.* 37.
6. Join B and C' forming $\triangle S$.	6. *Prin.* 13.
7. $\dfrac{\triangle T}{\triangle S} = \dfrac{b}{b'}$. $\dfrac{\triangle S}{\triangle T'} = \dfrac{c}{c'}$.	7. § 244.
8. $\dfrac{\triangle T}{\triangle S} \cdot \dfrac{\triangle S}{\triangle T'} = \dfrac{b}{b'} \cdot \dfrac{c}{c'}$ or $\dfrac{\triangle T}{\triangle T'} = \dfrac{bc}{b'c'}$.	8. Axiom of Multiplication.

262. Tool XX. Triangles with an angle of one equal to an angle of another have the same ratio as the products of the sides including the equal angles.

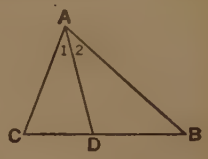
Exercises. Ratios of Products of Lines

1. In the annexed figure, if $\angle 1 = \angle 2$, by the use of Tool XX write:

a) the ratio of $\triangle ADC$ to $\triangle ADB$;

b) the ratio of $\triangle ACD$ to $\triangle ACB$ using $\angle C$ as the common angle;

c) the ratio of $\triangle ABD$ to $\triangle ABC$ using $\angle B$ as the common angle.

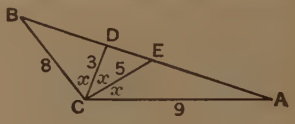


2. In the annexed figure, if x has the same value in each triangle, find by Tool XX the ratio of:

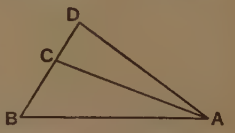
a) $\triangle BCD$ to $\triangle CDE$;

b) $\triangle CDE$ to $\triangle ACE$;

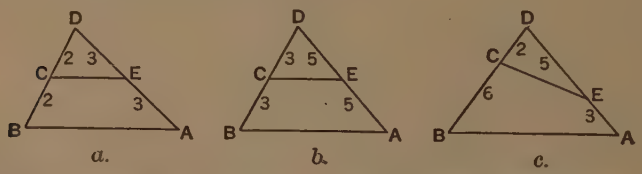
c) $\triangle BCD$ to $\triangle ACE$.



3. Find the ratio of $\triangle ADC$ to $\triangle ADB$, if $CD = \frac{1}{2}\,BD$; if $CD = \frac{1}{3}\,BD$; if $CD = \frac{3}{4}\,BD$.



4. Find the ratio of $\triangle ABD$ to $\triangle ABC$, if $CD = \frac{1}{2}\,BD$; if $CD = \frac{1}{3}\,BD$; if $CD = \frac{3}{4}\,BD$.



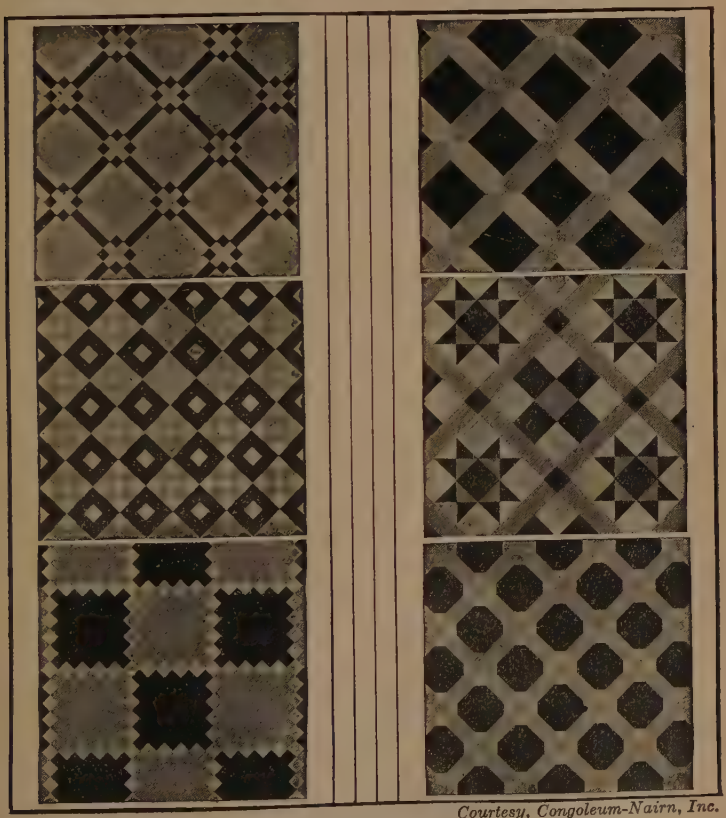
5. Find the ratio of $\triangle CDE$ to $\triangle BDA$ in each of the figures *a*, *b*, and *c*.

Proportions

263. Proportion. An equation whose members are *equal ratios* is called a *proportion*. Many of the expressions previously used were proportions without being thus designated. For example, the equality, $\frac{\triangle 1}{\triangle 2} = \frac{a}{a'}$, is a proportion. A pro-

portion may be written either in the form $\frac{a}{b} = \frac{c}{d}$ or in the form $a:b = c:d$, and both are read, "a is to b as c is to d." The two forms have the same meaning and either may be used in proofs and calculations.

264. Named Parts of a Proportion. In order to study the relations between the parts of a proportion it is convenient



Courtesy, Congoleum-Nairn, Inc.

FLOOR COVERINGS

The designs of linoleum, tile-work, and parquet flooring are frequently composed of geometric figures.

to employ special names. In the proportion $a : b = c : d$, the parts a, b, c, and d are called the *terms* of the proportion. The first and the fourth terms are called *extremes;* the second and the third terms are called *means*. The means of a proportion may be equal as in the *mean proportion* $a : b = b : c$. In this case b is called a *mean proportional* between a and c. Since such a proportion has only three different terms, c is called a *third proportional* to a and b. When the four terms of a proportion are all different, as in $\frac{a}{b} = \frac{c}{d}$; the fourth term is called a *fourth proportional* to the first three terms. The first terms of the ratios (i.e., the numerators of the fractions) are called the *antecedents;* the second terms of the ratios (i.e., the denominators of the fractions) are called the *consequents*. The literal meaning of antecedents is the terms that *go before;* and of consequents, the terms that *follow*.

265. A Proportion as a Fractional Equation. A proportion with its ratios in the fractional form (as $\frac{a}{b} = \frac{c}{d}$) is equivalent to a *fractional equation* and all the operations which may be performed upon a fractional equation may also be performed upon a proportion.

Oral Exercises

1. How many terms are in every simple proportion?

2. If the ratios of a proportion are not equal to unity, how many terms may be alike? Illustrate.

3. If the second and the third terms of a proportion are alike, what is the "like term" called?

4. In the proportion, $p : r = s : t$, name the extremes; the means; the antecedents; the consequents.

5. Does the meaning of the term "extremes" in a proportion suggest any similar meaning of the same word outside of mathematics? Consider the word "means" in the same way.

6. If the first ratio of a proportion is $\frac{1}{2}$, name several fractions which might form the second ratio.

7. Why is $\frac{3}{5} = \frac{5}{15}$ a true proportion? Is $\frac{6}{8} = \frac{4}{6}$ a true proportion? Why?

8. In algebra, what operation is performed upon a fractional equation in order to get rid of the denominators? May a proportion be treated in the same manner?

9. Solve for x in the proportion, $\frac{x}{4} = \frac{3}{6}$.

HINT: Multiply each member by 12.

10. Solve for x in the proportion, $\frac{7}{8} = \frac{2}{x}$.

11. Solve for y in the proportion, $\frac{y}{a} = \frac{b}{c}$.

12. Solve for x in the proportion, $\frac{a}{x} = \frac{x}{b}$.

NOTE: The proofs of Propositions 7–11 which follow may be omitted at the option of the teacher.

Proposition 7

266. Theorem. *The product of the extremes of a proportion equals the product of the means.*

Given: $\frac{a}{b} = \frac{c}{d}$.

To prove: $ad = bc$.

Proof:

STATEMENTS	FACTS
1. $\frac{a}{b} = \frac{c}{d}$.	1. Given.
2. $\frac{a}{b} \cdot bd = \frac{c}{d} \cdot bd$.	2. Axiom of Multiplication. (The multiplier is the *L. C. M.* of b and d.)
3. $ad = bc$.	3. By reduction.

267. Corollary 1. *A mean proportional between two quantities equals the square root of their product.*

Given: $\frac{a}{b} = \frac{b}{c}$.

To prove: $b = \sqrt{ac}$.

Proof: STATEMENTS	FACTS
1. $\frac{a}{b} = \frac{b}{c}$.	1. Given.
2. $b^2 = ac$.	2. § 266.
3. $b = \sqrt{ac}$.	3. *Prin.* 9.

268. Corollary 2. *If the product of two quantities equals the product of two other quantities, the quantities forming one product may be made the extremes and the quantities forming the other product may be made the means of a proportion.*

Given: $de = gh$.

To prove: $\frac{d}{g} = \frac{h}{e}$.

Proof: STATEMENTS	FACTS
1. $de = gh$.	1. Given.
2. $\frac{de}{ge} = \frac{gh}{ge}$.	2. Axiom of Division. (The divisor is the *L. C. M.* of the denominators of the conclusion.)
3. $\frac{d}{g} = \frac{h}{e}$.	3. By reduction.

269. Corollary 3. *If the numerators of a proportion are equal the denominators are equal, and conversely.*

270. Corollary 4. *If two proportions are alike except for two corresponding terms, those terms are equal.*

271. Tool XXI. The product of the extremes of a proportion equals the product of the means.

This Tool is used in many of the proofs and calculations involving proportions. It is the easiest method of testing a proportion to discover whether it is true or false. It is used to change an equality of two ratios into an equality of two products. It is also used in solving for an unknown quantity in a proportion.

Oral Exercises. Proportions

1. Test the following to determine which are true proportions: (*a*) $5:10 = 6:12$; (*b*) $4:10 = 6:9$; (*c*) $8:6 = 4:3$; (*d*) $20:5 = 16:4$; (*e*) $12:8 = 4:5$. (Use Tool XXI.)

2. Find the value of x in the following proportions: (*a*) $x : 4 = 12 : 8$; (*b*) $16 : x = 24 : 3$; (*c*) $4 : 3 = x : 12$; (*d*) $5 : 20 = 3 : x$; (*e*) $4 : x = x : 9$; (*f*) $2 : x + 1 = x - 1 : 4$.

3. Find a fourth proportional to (*a*) 2, 3, and 4; (*b*) 4, 6, and 8; (*c*) 8, 6, and 4.

HINT: $\frac{2}{3} = \frac{4}{x}$.

4. Find a mean proportional between (*a*) 1 and 4; (*b*) 4 and 9; (*c*) 9 and 16; (*d*) 16 and 9.

HINT: $x = \sqrt{1 \times 4}$ (§ 267).

5. Find a third proportional to (*a*) 2 and 4; (*b*) 3 and 6; (*c*) 6 and 3; (*d*) 4 and 2.

HINT: $\frac{2}{4} = \frac{4}{x}$.

6. From the equality $de = gh$ prove:

(*a*) $\frac{g}{e} = \frac{d}{h}$; (*b*) $\frac{h}{d} = \frac{e}{g}$; (*c*) $\frac{e}{g} = \frac{h}{d}$.

7. From the equality $3 \times 10 = 5 \times 6$ write:

(*a*) two proportions beginning with 3;
(*b*) two proportions beginning with 5;
(*c*) two proportions beginning with 10;
(*d*) two proportions beginning with 6.
Check each result by Tool XXI.

8. If $\frac{3}{x} = \frac{3}{y}$, what conclusion follows? Why?

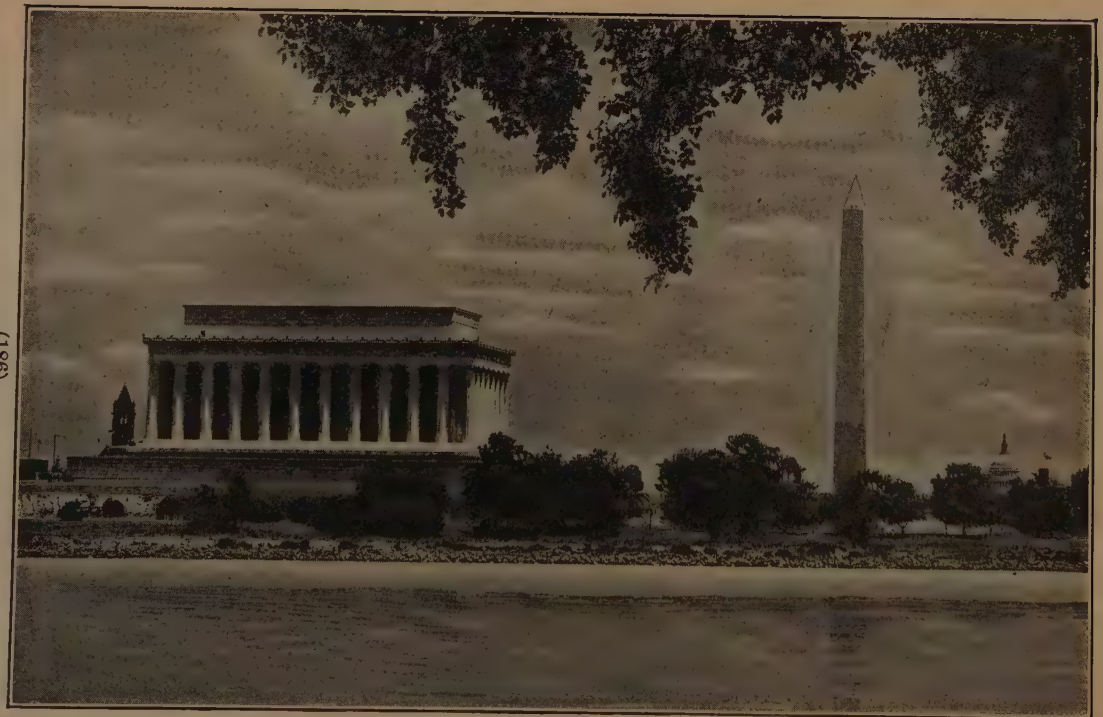
If $\frac{a}{x} = \frac{b}{x}$, what conclusion follows? Why?

If $\frac{z}{n} = \frac{z}{n'}$, what conclusion follows? Why?

9. Find the ratio of x to y from each of the following equalities: (*a*) $3x = 4y$; (*b*) $2x = 4y$; (*c*) $3x = y$; (*d*) $\frac{1}{2}x = y$; (*e*) $x = \frac{1}{3}y$; (*f*) $\frac{1}{2}x = \frac{1}{3}y$; (*g*) $ax = by$; (*h*) $ax = b(y - 2x)$.

10. If $\frac{2}{3} = \frac{x}{4}$ and $\frac{2}{3} = \frac{x'}{4}$, what conclusion follows? Why?

11. If $a : b = c : d$, does $ad = bc$? Why? If $a : c = b : d$, does $ad = bc$? Why? If $b : a = d : c$, does $ad = bc$? Why? If $c : a = d : b$, does $ad = bc$? Why?



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THE LINCOLN MEMORIAL AND WASHINGTON MONUMENT

The Lincoln Memorial, the Washington Monument, and the Capitol offer a contrast in architectural designs due to geometric forms. These three structures are on the main axis of the Mall; that is, on the East-West line described in connection with the map shown on page 157.

The pupil should note that there are four different proportional relations expressed in Ex. 11, p. 185, and only one product relation of the quantities a, b, c, and d.

TRANSFORMATION THEOREMS

Proposition 8

272. Theorem. *Four quantities in proportion are in proportion also by* ***alternation;*** *that is, the first term is to the third as the second is to the fourth.*

Given: $a : b = c : d$.

To prove: $a : c = b : d$.

Proof:

STATEMENTS	FACTS
1. $a : b = c : d$.	1. Given.
2. $ad = bc$.	2. § 266.
3. $a : c = b : d$.	3. § 268.

Proposition 9

273. Theorem. *Four quantities in proportion are in proportion also by* ***inversion;*** *that is, the second term is to the first as the fourth is to the third.*

Given: $a : b = c : d$.

To prove: $b : a = d : c$.

(To be completed by the pupil.)

Oral Exercises

1. State the following proportions by alternation and check:
(*a*) $5 : 10 = 6 : 12$; (*b*) $8 : 6 = 4 : 3$; (*c*) $20 : 5 = 16 : 4$.

2. State the following proportions by inversion and check:
(*a*) $4 : 8 = 6 : 12$; (*b*) $4 : 6 = 6 : 9$; (*c*) $5 : 20 = 3 : 12$.

3. Separate each of the following fractions into two terms, fractional or integral: (*a*) $\frac{a+b}{c}$; (*b*) $\frac{g+h}{t}$; (*c*) $\frac{a+b}{b}$; (*d*) $\frac{a-b}{b}$.

HINT: $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.

4. What operation is performed upon $\frac{a}{b}$ to obtain $\frac{a+b}{b}$?

5. What operation is performed upon $\frac{a}{b}$ to obtain $\frac{a-b}{b}$?

Proposition 10

274. Theorem. *Four quantities in proportion are in proportion also by* ***addition;*** *that is, the sum of the first two terms is to the second as the sum of the last two is to the fourth.*

Given: $a : b = c : d$.

To prove: $\frac{a+b}{b} = \frac{c+d}{d}$.

Proof:

STATEMENTS	FACTS
1. $\frac{a}{b} = \frac{c}{d}$.	1. Given.
2. $\frac{a}{b} + 1 = \frac{c}{d} + 1$.	2. Axiom of Addition.
3. $\frac{a}{b} + \frac{b}{b} = \frac{c}{d} + \frac{d}{d}$, or $\frac{a+b}{b} = \frac{c+d}{d}$.	3. By reduction.

275. Corollary. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a} = \frac{c+d}{c}$.

HINT: Use Inversion and Addition.

Proposition 11

276. Theorem. *Four quantities in proportion are in proportion also by* ***subtraction;*** *that is, the first term minus the second is to the second as the third minus the fourth is to the fourth.*

Given: $a : b = c : d$.

To prove: $\frac{a-b}{b} = \frac{c-d}{d}$.

(To be completed by the pupil.)

277. Corollary. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{a} = \frac{c-d}{c}$.

HINT: $1 - \frac{b}{a} = \frac{a-b}{a}$.

Exercises

1. Write the following proportions by addition and check:
(*a*) $12:6 = 10:5$; (*b*) $4:3 = 8:6$; (*c*) $20:5 = 16:4$.

2. Write the proportions of Ex. 1 by subtraction. Check.

3. If $\frac{p+r}{r} = \frac{s+t}{t}$, does it follow that $\frac{p}{r} = \frac{s}{t}$? Why?

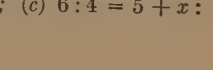
4. If $\frac{p-r}{r} = \frac{s-t}{t}$, does it follow that $\frac{p}{r} = \frac{s}{t}$? Why?

5. Write the proportion, $12:8 = 9:6$, (*a*) by alternation; (*b*) by inversion; (*c*) by addition; (*d*) by subtraction. Check each transformed proportion.

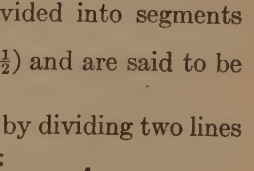
6. Transform the following proportions so that only one term contains x: (*a*) $4:6 = 5 - x:x$; (*b*) $8:6 = 2 + x:x$; (*c*) $6:4 = 5 + x:5 - x$; (*d*) $3 + x:3 - x = 5:3$.

HINT: Use §§ 274 and 276.

278. Line Segments in Proportion. Two lines are said to be *divided proportionally* when the segments of the one line have the same ratio as the corresponding segments of the other. For example, if p and r, the segments of line m, are so related that $\frac{p}{r} = \frac{2}{4} = \frac{1}{2}$; and if s and t, the segments of line n, are so related that $\frac{s}{t} = \frac{3}{6} = \frac{1}{2}$; then m and n are divided into segments which have the same ratio (in this case $\frac{1}{2}$) and are said to be divided *proportionally*.



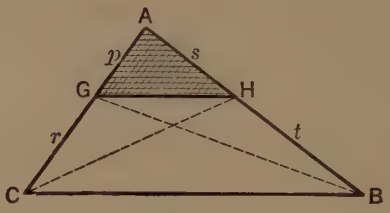
A second illustration of what is meant by dividing two lines proportionally is given in the following: the sides AC and AB of $\triangle ABC$ are divided proportionally by the line GH, if $\frac{p}{r} = \frac{s}{t}$.



In a proportion involving line segments, the ratios are understood to be the ratios of the numerical measures of those segments.

Proposition 12 (c N)

279. Theorem. *A line parallel to a side of a triangle divides the other two sides proportionally.*



Given: $GH \parallel CB$ in $\triangle ABC$.

To prove: $\frac{p}{r} = \frac{s}{t}$.

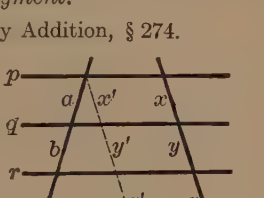
HINT: May the terms of each ratio ($\frac{p}{r}$ and $\frac{s}{t}$) be taken as the bases of triangles which have a common altitude? (See Part 2, § 260.)

Proof: STATEMENTS	FACTS
1. Draw CH and BG.	1. *Prin.* 13.
Since $\triangle AGH$ and $\triangle CGH$ have a common altitude from H,	
2. $\frac{p}{r} = \frac{\triangle AGH}{\triangle CGH}$.	2. § 244.
Since $\triangle AGH$ and $\triangle BGH$ have a common altitude from G,	
3. $\frac{s}{t} = \frac{\triangle AGH}{\triangle BGH}$.	3. Why?
4. $GH \parallel CB$.	4. Why?
5. $\triangle CGH = \triangle BGH$.	5. § 246.
6. $\frac{p}{r} = \frac{\triangle AGH}{\triangle CGH} = \frac{\triangle AGH}{\triangle BGH} = \frac{s}{t}$.	6. Sub.

280. Corollary 1. *One side of a triangle and either segment cut off by a line parallel to a second side have the same ratio as the third side and its corresponding segment.*

HINT: Transform the conclusion of § 279 by Addition, § 274.

281. Corollary 2 (c N). *The segments cut off on two transversals by a series of parallel lines are proportional.*



Given: Two transversals with $p \parallel q \parallel r \parallel s$.

To prove: $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$.

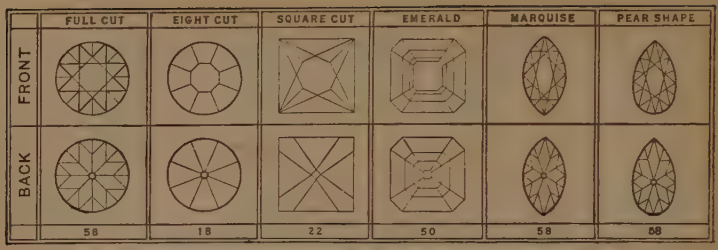
Proof:

STATEMENTS	FACTS
1. Through the extremity of a, draw $d \parallel e$.	1. *Prin.* 14.
2. $x' = x$, $y' = y$, and $z' = z$.	2. § 147.
3. $\frac{a+b}{x'+y'} = \frac{a}{x'} = \frac{b}{y'}$.	3. § 280 and § 272.
4. $\frac{a+b}{x'+y'} = \frac{c}{z'}$.	4. § 279.
5. $\frac{a}{x'} = \frac{b}{y'} = \frac{c}{z'}$.	5. Sub.
6. $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$.	6. Sub.

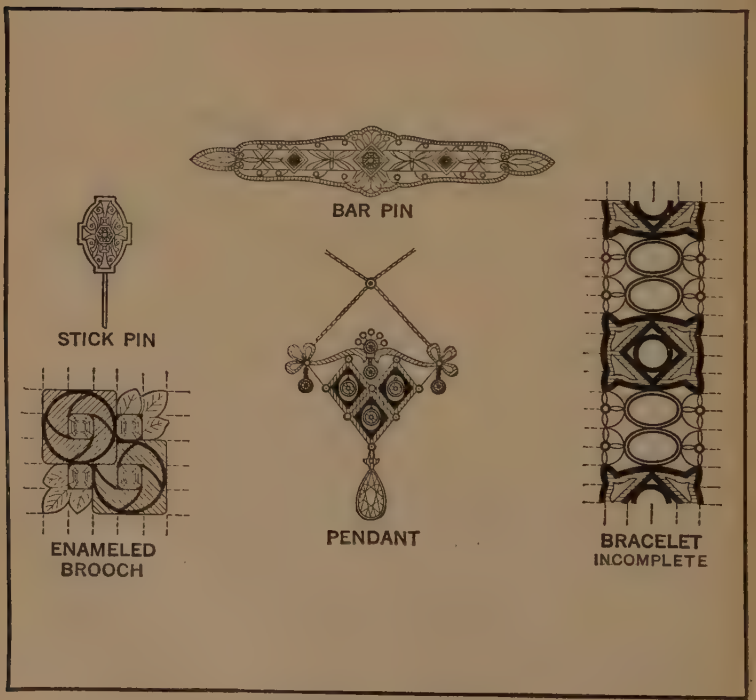
282. Corollary 3 (C N). *If three or more parallel lines intercept equal segments on one transversal, they intercept equal segments on every transversal.*

Proof: In the figure of § 281,

1. $a = b = c$. Given.
2. $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$. (§ 281.)
3. $x = y = z$. (§ 269.)



GEOMETRIC FORMS IN PRECIOUS STONES
(The small figures show the total number of faces or "facets.")

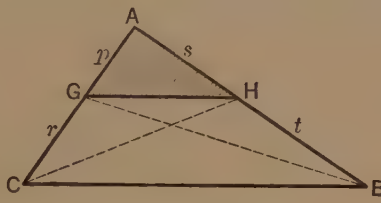


GEOMETRY IN JEWELRY

283. Tool XXII. One side of a triangle and either segment cut off by a line parallel to a second side have the same ratio as the third side and its corresponding segment.

Proposition 13 (C N)

284. Theorem. *A line which divides two sides of a triangle proportionally is parallel to the third side.* (Converse of § 279.)



Given: GH in $\triangle ABC$ making $\frac{p}{r} = \frac{s}{t}$.

To prove: $GH \parallel BC$.

HINT: (See Hint given for Proposition 12.)

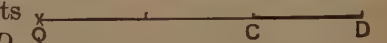
Proof:

STATEMENTS	FACTS
1. Draw CH and BG.	1. *Prin.* 13.
2. $\frac{\triangle AGH}{\triangle CGH} = \frac{p}{r}$.	2. § 244.
3. $\frac{\triangle AGH}{\triangle BGH} = \frac{s}{t}$.	3. § 244.
4. $\frac{p}{r} = \frac{s}{t}$	4. Given.
5. $\frac{\triangle AGH}{\triangle CGH} = \frac{\triangle AGH}{\triangle BGH}$.	5. Sub.
6. $\triangle CGH = \triangle BGH$.	6. § 269.
7. $GH \parallel BC$.	7. § 247.

285. A Line Divided Internally—Externally—Harmonically.

The point P on the line AB divides the line *internally* into two segments which have the ratio $PA : PB$.

The point Q on the line CD *produced* divides the line *externally* into two segments which have the ratio $QC : QD$.



In each case, the segments lie between the point of division and the extremities of the line.

If point P on the line EF and the point Q on EF produced divide the line *internally* and *externally* into segments which have equal ratios so that $PE : PF = QE : QF$, the line is said to be divided *harmonically*.

Oral Exercises

1. Divide a line 10 in. long into two segments which have the ratio 2 : 3.

HINT: $2x + 3x = 10$ in.

2. Divide a 24-inch line internally into segments which have the following ratios: (*a*) 1 : 2; (*b*) 1 : 3; (*c*) 3 : 5; (*d*) 5 : 7.

3. Divide a 10-inch line externally into segments which have the ratio 2 : 3.

HINT: $3x - 2x = 10$ in.

4. Divide a 24-inch line externally into segments which have the ratios: (*a*) 1 : 2; (*b*) 1 : 3; (*c*) 3 : 5.

5. Divide a 24-inch line harmonically in the ratio: (*a*) 1 : 2; (*b*) 3 : 5.

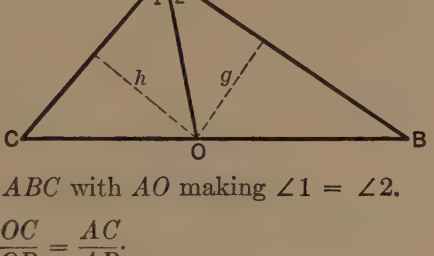
6. In the figure of § 279, if $3p = 2r = 6$ and $s = 3$, how long is t? how long is AB?

7. In the figure of § 281, if $4a = 6b = 3c = 12$ and $z = 6$, find x and y.

8. Which theorems thus far studied in Chapter Three have given rise to ratios in triangles?

Proposition 14 (c N)

286. Theorem. *The bisector of an interior angle of a triangle divides the opposite side into segments which have the same ratio as the other two sides.*



Given: $\triangle ABC$ with AO making $\angle 1 = \angle 2$.

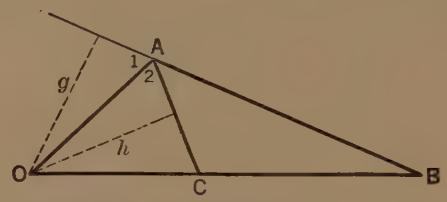
To prove: $\frac{OC}{OB} = \frac{AC}{AB}$.

HINT: What is the relation of triangles with the same or equal altitudes? What Fact relates to a point on the bisector of an angle?

Proof: STATEMENTS	FACTS
1. $\frac{OC}{OB} = \frac{\triangle ACO}{\triangle ABO}$.	1. Why?
2. Draw $\perp$s g and h.	2. By what Principle?
3. $\angle 1 = \angle 2$.	3. Why?
4. $g = h$.	4. § 172.
Since h is the altitude of $\triangle ACO$ with base AC and g is the altitude of $\triangle ABO$ with base AB,	
5. $\frac{\triangle ACO}{\triangle ABO} = \frac{AC}{AB}$.	5. Why?
6. $\frac{OC}{OB} = \frac{\triangle ACO}{\triangle ABO} = \frac{AC}{AB}$.	6. Why?

Proposition 15 (c N)

287. Theorem. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments which have the same ratio as the other two sides.*

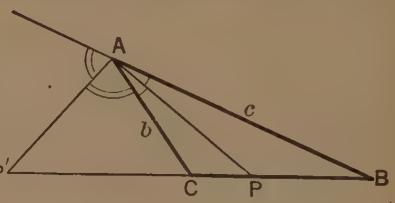


Given: $\triangle ABC$ with AO making $\angle 1 = \angle 2$.

To prove: $\dfrac{OC}{OB} = \dfrac{AC}{AB}$.

HINT: The proof is identical to that of Proposition 14 and should be completed by the pupil.

288. Corollary. *The bisector of an interior angle and the bisector of an exterior angle at the same vertex of a triangle divide the opposite side harmonically in the same ratio as the other two sides.*

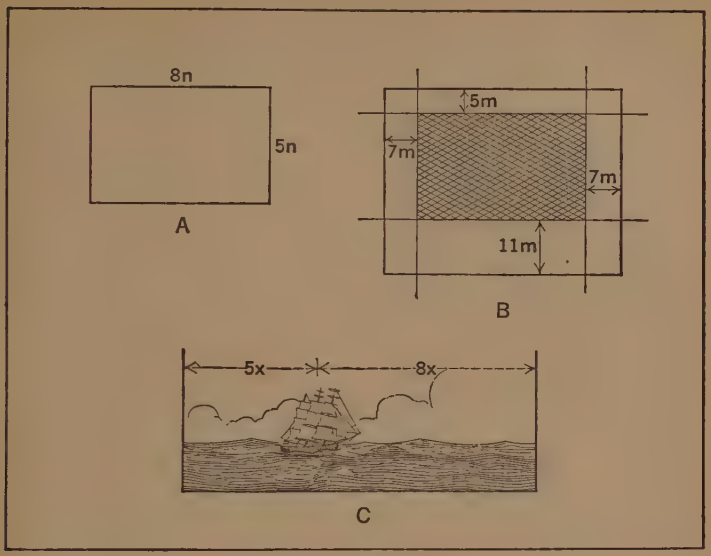


Proof: $\dfrac{PC}{PB} = \dfrac{b}{c}$ by § 286.

$\dfrac{P'C}{P'B} = \dfrac{b}{c}$ by § 287.

$\dfrac{PC}{PB} = \dfrac{b}{c} = \dfrac{P'C}{P'B}$ by Substitution.

Exercise. In the figure of § 288, if $b = 9$, $c = 15$, and $CB = 8$, find PC, PB, $P'C$, and $P'B$. What is the value of the ratio $BC : PP'$? How are PA and $P'A$ related? Why?

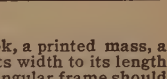


RATIOS IN ADVERTISING

Fig. *A*. A rectangular form like a sheet of paper, cover of a book, a printed mass, a sketch, or photograph has the greatest appeal when the ratio of its width to its length is approximately 5 : 8. Fig. *B*. Rectangular masses within a rectangular frame should have margins in the following ratio: top, sides, bottom = 5 : 7 : 11. Fig. *C*. The important object of an illustration or photograph should be placed at a point whose distances from the sides have a ratio of 5 : 8. (Other ratios which may be used are 2 : 3, 3 : 5, and 8 : 13.)

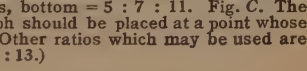
Exercises. Ratios of Line Segments

1. In the annexed figure, if $\angle 1 = \angle 2$ write a proportion beginning with (*a*) 6; (*b*) 9; (*c*) 8.



2. What is the ratio of the two smaller triangles in Ex. 1? Why? Can you obtain the result in two ways?

3. In the annexed figure, if $\angle 1 = \angle 2$, write a proportion beginning with (*a*) 20; (*b*) 6; (*c*) 30.



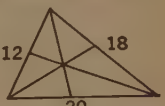
4. In the annexed figure, $\angle 1 = \angle 2$. Find the segments of side 15.

HINT: The segments are in the ratio of 9 : 12 or 3 : 4. Then $3x + 4x = 15$.

5. In the same figure, if $\angle B$ is bisected, find the segments of side 9.

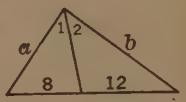
6. In the same figure, if $\angle C$ is bisected, find the segments of side 12.

7. If each angle of the annexed figure is bisected by a line which terminates in the opposite side, what is the ratio of the segments into which (*a*) side 20 is divided? (*b*) side 12? (*c*) side 18?

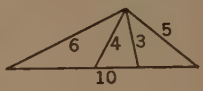


8. In the figure of Ex. 7 and under the same conditions, what is the ratio of the segments of the bisector drawn to line 20?

9. In the annexed figure, if $\angle 1 = \angle 2$, what are some of the possible values of a and b greater than 8 and 12?

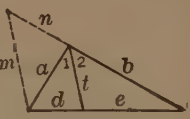


10. In the annexed figure, if the vertex angle is trisected, find the segments of side 10.



11. In the annexed figure, $\frac{d}{e} = \frac{a}{b}$. Prove $\angle 1 = \angle 2$.

HINT: Draw $m \parallel t$ and produce b until it meets m.



12. In the figure of Ex. 11, if $\angle 1 = \angle 2$ and $m \parallel t$, assign values to b, n, and e and find d; assign values to b, e, and d and find a; prove $n = a$.

(Supplementary Exercises 35–42, p. 202, may be studied now.)

SUPPLEMENTARY EXERCISES

Exercises. Areas of Rectangles

1. Find the dimensions of a rectangle whose area is 96 sq. in. and whose length is (*a*) 4 in. more than its width; (*b*) 10 in. more than its width.

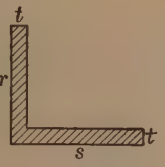
2. The length of a rectangle is 3 in. more than twice the width. Find the area when its perimeter is (*a*) 24 in.; (*b*) 30 in.

3. A fence whose entire length is 80 ft. encloses a rectangular yard 15 ft. wide. How much more surface would it surround if the yard were square?

4. A picture $10'' \times 15''$ is surrounded by a frame of uniform width whose area is the same as the area of the picture. Find the width of the frame.

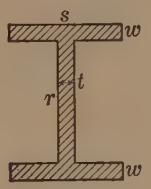
5. Find the number of square inches in the cross section of the L-beam with the following dimensions.

	r	s	t
a	5	4	$\frac{1}{4}$
b	10	7	$\frac{1}{2}$
c	4	3	$\frac{1}{4}$

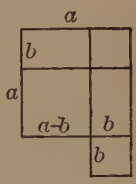


6. Find the number of square inches in the cross section of the I-beam with the following dimensions:

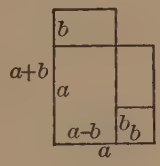
	r	s	t	w
a	8	6	$\frac{1}{2}$	$\frac{1}{4}$
b	10	8	$\frac{3}{4}$	$\frac{1}{2}$
c	12	10	$\frac{3}{4}$	$\frac{1}{2}$



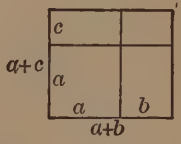
7. Show geometrically that $(a - b)^2 = a^2 - 2ab + b^2$.



8. Show geometrically that $(a + b)(a - b) = a^2 - b^2$



9. Show geometrically that $(a + b)(a + c) = a^2 + a(b + c) + bc$.

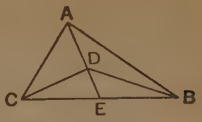


Exercises. Areas of Triangles

10. If c is the hypotenuse of an isosceles right triangle, the area is $\frac{c^2}{4}$. Prove.

11. In the $\square$ *PRST* prove $\triangle$ *PRS* = $\triangle$ *TRS*.

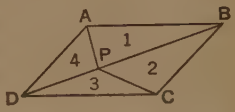
12. In the figure, AE is a median. Prove $\triangle ADC = \triangle ABD$.



HINT: Use Axiom of Subtraction.

13. Find the area of the annexed figure in terms of a and b.

14. In the annexed figure, BD is the diagonal of $\square ABCD$ and P, any point on the diagonal, is joined to A and C. Prove $\triangle 1 = \triangle 2$ and $\triangle 3 = \triangle 4$.



15. The sides of a right triangle are in the ratio of 3 : 4 : 5. If the area is 150, find the altitude upon the hypotenuse.

HINT: Let $3x$, $4x$, and $5x$ be the sides.

16. The area of a triangle equals one-half the product of its perimeter by the radius of the inscribed circle.

17. If the quadrilateral $ABCD$ has $\triangle ABC = \triangle ADC$, prove that AC bisects BD.

18. If two triangles have two sides of the one equal respectively to two sides of the other and the included angles supplementary, the triangles are equal.

HINT: Apply one triangle to the other so that the exterior sides of the supplementary angles lie in a straight line.

Exercises. Areas of Parallelograms

19. A rhombus contains 100 sq. ft. and the shorter diagonal is 10 ft. Find the length of the longer diagonal.

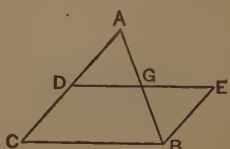
20. The square upon the diagonal of a square is double the original square.

HINT: Let s equal the side of the original square and find the diagonal in terms of s.

21. The diagonals of a rhombus are 18 and 4. Find the side of an equal square.

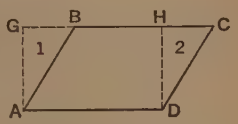
HINT: Use § 248.

22. In the annexed figure, $BCDE$ is a $\square$ with G the mid-point of DE. Prove $\triangle ABC = \square BCDE$.



23. A line bisecting the opposite sides of a parallelogram bisects the parallelogram.

24. By using the annexed figure, prove that the area of a parallelogram equals the product of a base by an altitude.



HINT: $\square ABCD = AGCD - \triangle 1$, and rectangle $AGHD = AGCD - \triangle 2$.

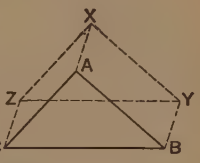
25. Through any point in the diagonal of a parallelogram lines are drawn parallel to the sides. Of the four parallelograms, which are formed, the two which do not contain the diagonal are equal.

26. One diagonal of a rhombus is $1\frac{2}{3}$ times the other and their difference is 8, find the area.

HINT: $3x = 5y$ and $x - y = 8$.

27. Any line through the mid-point of a diagonal of a parallelogram divides the parallelogram into two equal parts.

28. In the annexed figure, the corresponding lines are parallel. Prove $\square AZ + \square AY = \square BZ$.



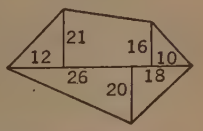
Exercises. Areas of Trapezoids

29. Find the value of x in the following trapezoids:

	AREA	ALTITUDE	UPPER BASE	LOWER BASE
a	x	6	8	10
b	x	10	7	13
c	42	x	6	8
d	88	8	x	12
e	90	9	6	x

30. A trapezoid contains 240 sq. in. and its altitude is 16 in. Find the two bases (*a*) if one is 3 in. longer than the other; (*b*) if they have the ratio of 2 : 3.

31. Find the area of the polygon shown in the annexed figure.



32. Through the mid-point of one leg of a trapezoid a line is drawn parallel to the other leg and meeting both bases. Prove that the parallelogram thus formed equals the trapezoid.

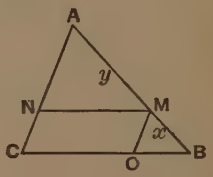
33. If a bisector of the median of a trapezoid meets the two bases, the trapezoid is divided into two equal parts.

34. Suggest a way for finding the approximate area of an irregular figure by means of trapezoids.

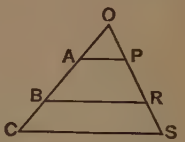
Exercises. Ratios of Line Segments

35. In the annexed figure, M is any point in AB of $\triangle ABC$, $MN \parallel BC$, and $MO \parallel AC$. Prove $\frac{BO}{OC} = \frac{CN}{NA}$.

HINT: Prove each ratio equal to $\frac{x}{y}$.



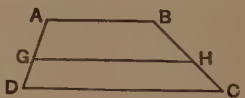
36. In the annexed figure, if $\frac{OA}{AC} = \frac{OP}{PS}$ and $\frac{OB}{BC} = \frac{OR}{RS}$. Prove $\frac{OB}{OA} = \frac{OR}{OP}$.



37. Prove by Proposition 13, § 284, that a line which bisects two sides of a triangle is parallel to the third side.

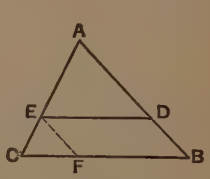
38. State and prove the converse of Proposition 15, § 287.

39. A line parallel to the bases of a trapezoid divides the non-parallel sides proportionally.



40. In $\triangle ABC$, $DE \parallel BC$. Prove $\frac{AE}{AC} = \frac{DE}{BC}$.

HINT. Draw $EF \parallel AB$.



41. If a pendulum is suspended from a point O and in any of its positions cuts two parallel lines a and b at points M and N respectively, show that the ratio $OM:ON$ is a constant.

42. From any point O without $\triangle ABC$ draw lines OA, OB, and OC. Bisect these lines by points A', B', and C' respectively. If the sides of $\triangle ABC$ are 7, 9, and 14, find the perimeter of $\triangle A'B'C'$.

SUMMARY OF CHAPTER THREE

The Area of a

1. **rectangle** = base × altitude;
2. **square** = square of a side; or half the square of a diagonal;
3. **parallelogram** = base × altitude;
4. **rhombus** = half the product of the diagonals;
5. **triangle** = half the product of an altitude by its base;
6. **trapezoid** = half the product of an altitude by the sum of the bases.

Proportions. (General Properties)

1. A proportion may be transformed by Alternation, Inversion, Addition, Subtraction, and by any combination of them.

2. In any proportion the product of the extremes equals the product of the means.

3. If the product of two quantities equals the product of a second pair, either pair may be made the means and the other pair the extremes of a proportion.

4. A mean proportional between two quantities equals the square root of their product.

5. If three terms of two proportions are respectively equal, the fourth terms are equal.

Proportions. (Ratios of Lines)

1. Rectangles, parallelograms, or triangles with equal $\left\{\begin{matrix}\text{altitudes}\\\text{bases}\end{matrix}\right\}$ have the same ratio as their $\left\{\begin{matrix}\text{bases}\\\text{altitudes}\end{matrix}\right\}$.

2. Triangles with an angle of one equal to an angle of another have the same ratio as the products of the including sides.

3. A line parallel to one side of a triangle divides the other two sides in the same ratio; and conversely.

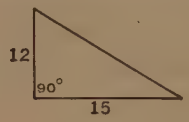
4. One side of a triangle and either segment cut off by a line parallel to a second side have the same ratio as the third side and its corresponding segment.

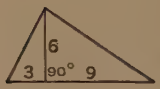
5. Corresponding segments cut off on two transversals by a series of parallel lines have the same ratio.

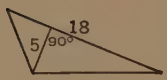
6. The bisector of an $\left\{\begin{matrix}\text{interior}\\\text{exterior}\end{matrix}\right\}$ angle of a triangle divides the opposite side $\left\{\begin{matrix}\text{internally}\\\text{externally}\end{matrix}\right\}$ in the same ratio as the other two sides.

REVIEW ORAL EXERCISES. DIAGNOSTIC TEST

1. Find the area of each of the following figures; state the Fact which supports each step of your work:

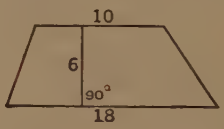
a. 

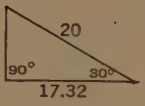
b. 

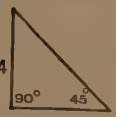
c. 

d.

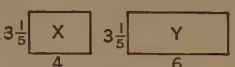
e.

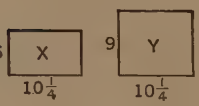
f. 

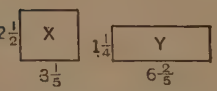
g. 

h. 

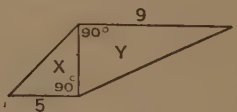
2. Find the ratio of the area of figure X to the area of figure Y in the following, and state in each case the theorem involved:

a. 

b. 

c. 

d.

e. 

f.

3. Show by drawing two triangles how it is possible for a triangle with a perimeter of 12 in. to have the same area as a triangle with a perimeter of 6 in.

4. Is it possible for two rectangles whose perimeters are 12 in. and 4 in. to have the same area? Is it true if the rectangles are squares? Why?

5. If a line bisects two consecutive sides of a parallelogram, the triangle formed is what part of the parallelogram?

6. May a square be divided into 2 squares? 4 squares? How?

7. Is any triangle divided into two equal triangles by a median? by an altitude? by the bisector of an angle?

8. What is the formula for the area of a triangle?

9. Does the perimeter of an equilateral triangle determine its area?

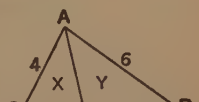
10. If a rectangle and a square have equal perimeters which is the larger in area?

11. What is the area of the largest quadrilateral that can be shaped from a wire 36 in. long?

12. The dimensions of a rectangle are 8 and 12. If its length is increased by 4, how much must be taken from its width to keep the area the same?

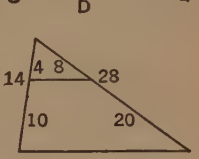
13. If two triangles have equal areas and the base of one is four times that of the other, what is the ratio of their altitudes?

14. If $p:q = r:s$ what is the transformed proportion which begins with p? with q? with $p + q$? with $p - q$?



15. In the annexed figure, if AD bisects $\angle A$ what is the ratio of the areas of triangles X and Y? Why is it unnecessary for AD to be known?

16. A line is parallel to the base of the triangle in the adjoining figure. What proportions may be written beginning with 4? with 20? with 14? with 8? with 28?



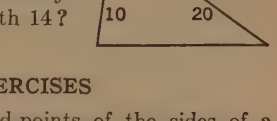
MISCELLANEOUS EXERCISES

1. The lines joining the successive mid-points of the sides of a quadrilateral form a parallelogram equal in area to half the quadrilateral.

2. If all the altitudes of a triangle are equal, the triangle is equilateral.

3. If $CE \parallel BD$, prove that quadrilateral $ABCD$ equals triangle ABE.

HINT: Prove $\triangle CBD = \triangle EBD$ and use Axiom of Addition.

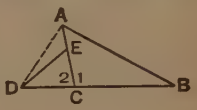


4. The area of a trapezoid equals the product of one of its legs and the perpendicular to that leg from the mid-point of the opposite side.

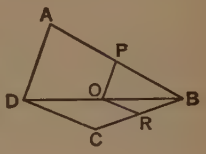
5. If two equal triangles have an angle of the one equal to an angle of the other, the products of the sides including the equal angles are equal.

6. Two equal triangles have a common base and lie on opposite sides of it. Prove that the base, produced if necessary, bisects the line joining their vertices.

7. If two triangles have an angle of the one supplementary to an angle of the other, the triangles have the same ratio as the products of the sides including those angles.



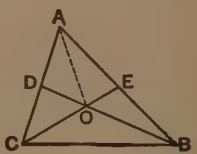
8. In the annexed figure, $OP \parallel DA$ and $OR \parallel DC$. Prove $PR \parallel AC$.



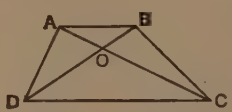
9. The altitude of a triangle is one-third of its base. Find the dimensions when the area is (*a*) 24; (*b*) 54; (*c*) 150.

10. If D and E are mid-points of the sides of the triangle in the annexed figure, prove that $\triangle BCO = ADOE$.

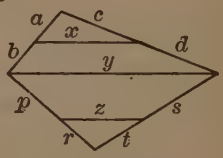
HINT: Draw AO.



11. In the trapezoid $ABCD$ the diagonals form three pairs of equal triangles. Find them and prove them equal.

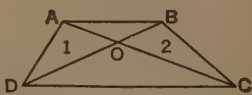


12. In the annexed figure, $\frac{a}{b} = \frac{c}{d}$ and $\frac{p}{r} = \frac{s}{t}$. Prove $x \parallel z$.

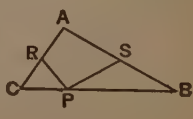


13. In the quadrilateral $ABCD$, if $\triangle 1 = \triangle 2$, prove $AB \parallel CD$.

HINT: Use Axiom of Addition and § 247.



14. If the mid-points of two sides of a triangle are joined to any point in the base, the quadrilateral thus formed equals half the triangle.



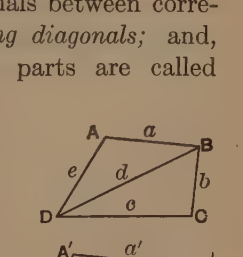
CHAPTER FOUR

SIMILAR POLYGONS—PROPORTIONAL MAGNITUDES

SIMILAR POLYGONS

289. Mutually Equiangular Polygons. If the angles of one polygon, taken in order, are equal respectively to those of another, taken in order, the polygons are said to be *mutually equiangular*. (This definition has already been given in § 162.)

290. Corresponding Parts. The pairs of equal angles in two mutually equiangular polygons, taken in order, are called *corresponding angles* and their vertices are called *corresponding vertices*. The sides between corresponding vertices are called *corresponding sides;* the diagonals between corresponding vertices are called *corresponding diagonals;* and, the triangles formed by corresponding parts are called *corresponding triangles*, or *like-placed triangles.*



If $ABCD$ and $A'B'C'D'$ are mutually equiangular, A and A', B and B′, C and C', etc., are corresponding angles; a and a', b and b', c and c', etc., are corresponding sides; d and d' are corresponding diagonals; $\triangle eda$ and $\triangle e'd'a'$ are corresponding triangles.

291. Proportional Sides—Ratio of Similitude. If the sides of one polygon taken in order as numerators, form a series of equal ratios with the sides of another polygon, taken in order as denominators, the polygons are said to have their

sides proportional. Thus in the previous figure, if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{e}{e'}$, the two polygons have their sides proportional; and any one of the ratios ($\frac{a}{a'}$, $\frac{b}{b'}$, etc.) is called the *ratio of similitude of the polygons.*

292. Similar Polygons. Two polygons are similar if

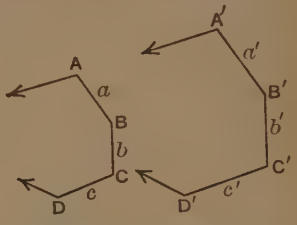
a) they are mutually equiangular *and*

b) their corresponding sides are proportional.

For example, in the figure, if we know that $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, etc., *and also*

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}, \text{ etc.,}$$

then we know that $ABCD \ldots .A$ and $A'B'C'D' \ldots .A'$ are similar.



If two polygons are known to be similar, it follows that **their corresponding angles are equal and their corresponding sides are proportional.**

The symbol of similarity is $\sim$ (the letter *S* written horizontally.) The expression $P \sim P'$ is read, "P is similar to P'."

Oral Exercises

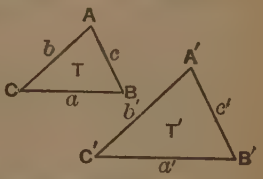
1. If two polygons are mutually equiangular, are they necessarily equiangular? If two polygons are equiangular, are they also mutually equiangular? Illustrate.

2. If two triangles are congruent, what is their ratio of similitude?

3. Why are equilateral triangles similar?

4. Why are squares similar?

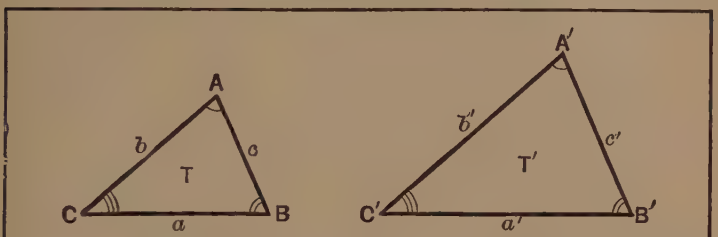
5. By Tool XX write the ratio of $\triangle T$ to $\triangle T'$ (*a*) if $\angle A = \angle A'$; (*b*) if $\angle B = \angle B'$; (*c*) if $\angle C = \angle C'$.



6. If $\triangle T$ and $\triangle T'$ are mutually equiangular, what else must be shown to prove them similar?

Proposition 1 (C)

293. Theorem. *Mutually equiangular triangles are similar.*



Given: $\triangle T$ and T', $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

To prove: $\triangle T \sim \triangle T'$.

HINT: What two conditions make polygons similar? What previous theorem gives the ratio of triangles with a pair of angles equal?

Proof: STATEMENTS	FACTS
1. $\angle A = \angle A'$, $\angle C = \angle C'$, $\angle B = \angle B'$.	1. Given.
2. $\frac{\triangle T}{\triangle T'} = \frac{bc}{b'c'}$, $\frac{\triangle T}{\triangle T'} = \frac{ab}{a'b'}$, $\frac{\triangle T}{\triangle T'} = \frac{ac}{a'c'}$.	2. § 261.
3. $\frac{\triangle T}{\triangle T'} = \frac{bc}{b'c'} = \frac{ab}{a'b'} = \frac{ac}{a'c'}$.	3. Sub.
4. $\frac{a}{a'} = \frac{c}{c'}$. (Dividing the second and third fractions by $\frac{b}{b'}$.)	4. *Prin.* 8.
5. $\frac{b}{b'} = \frac{c}{c'}$. (Dividing the third and fourth fractions by $\frac{a}{a'}$.)	5. *Prin.* 8.
6. $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.	6. Sub.
7. $\triangle T \sim \triangle T'$.	7. § 292.

294. Corollary 1 (C N). *Triangles with two angles of one equal to two angles of another are similar.* (*a.a.* = *a.a.*)

Proof: $\angle A = A'$ and $\angle B = \angle B'$ (Given).
$\angle C = \angle C'$ (§ 111).
$\triangle T \sim \triangle T'$ (§ 293).

295. Corollary 2. *Right triangles with an acute angle of one equal to an acute angle of another are similar.*

296. Corollary 3. *Triangles with parallel sides are similar.*

297. Corollary 4. *Triangles with perpendicular sides are similar.*

298. Corollary 5. *Triangles similar to the same triangle are similar to each other.*

299. Tool XXIII. Triangles with two angles of one equal respectively to two angles of another are similar.

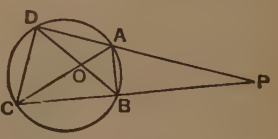
Exercises. Similar Triangles. (Use Tool XXIII.)

1. A line parallel to one side of a triangle and intersecting the other two sides cuts off another triangle similar to the first.

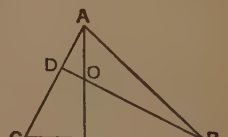
2. In the annexed figure, *GH* and *PR* are both perpendicular to the base of an isosceles triangle. Prove rt. $\triangle CHG \sim$ rt. $\triangle BRP$.

3. Two isosceles triangles are similar if a base angle of the one equals a base angle of the other.

4. In the annexed figure, *PD* and *PC* are secants. Prove $\triangle PCA \sim \triangle PDB$.



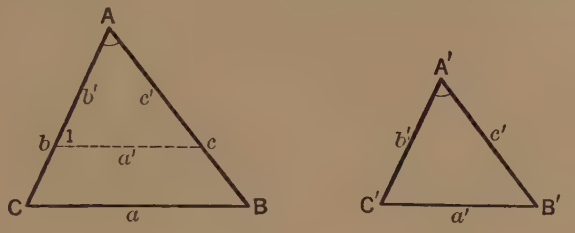
5. In the annexed figure, $AE \perp BC$ and $BD \perp AC$. Prove $\triangle AEC \sim \triangle ADO$.



(Supplementary Exercises 1–10, pp. 254, 255, may be studied now.)

Proposition 2 (C N)

300. Theorem. *Triangles with an angle of one equal to an angle of another and the including sides proportional are similar.*



Given: $\triangle ABC$ and $A'B'C'$, $\angle A = \angle A'$, $\frac{b}{b'} = \frac{c}{c'}$.

To prove: $\triangle ABC \sim \triangle A'B'C'$.

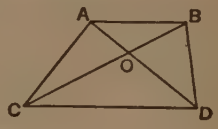
HINT: Place $\triangle A'B'C'$ upon $\triangle ABC$. What condition makes two triangles similar? (See Tool XXIII.)

Proof: STATEMENTS	FACTS
1. $\angle A = \angle A'$.	1. Given.
2. $\angle A$ can be made to coincide with $\angle A'$.	2. § 24.
3. Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle A'$ coincides with its equal $\angle A$.	3. *Prin.* 37.
4. $\frac{b}{b'} = \frac{c}{c'}$.	4. Given.
5. $\frac{b - b'}{b'} = \frac{c - c'}{c'}$.	5. § 276.
6. $a' \parallel a$.	6. § 284.
7. $\angle 1 = \angle C$.	7. Why?
8. $\triangle ABC \sim \triangle A'B'C'$.	8. Why?

Exercises. Similar Triangles

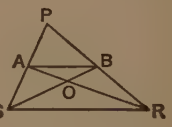
1. Two right triangles are similar if two legs of the one are proportional to the two legs of the other.

2. In the annexed figure, if $AB \parallel CD$, prove $\triangle AOB \sim \triangle DOC$.



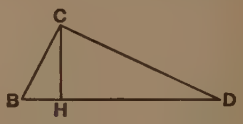
3. The diagonals of an isosceles trapezoid form two pairs of similar triangles.

4. In the annexed figure, RA and SB are medians. Prove $\triangle AOB \sim \triangle ROS$ and $\triangle PAB \sim \triangle PSR$.



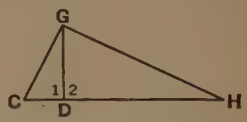
5. If two triangles are similar, they are divided into pairs of similar triangles by two corresponding medians.

6. In the figure, BC is the mean proportional between BH and BD. Prove $\triangle BHC \sim \triangle BCD$.



7. In the figure of Ex. 6, if $\angle BCH = \angle D$, prove $\triangle BCH \sim \triangle BDC$.

8. In the figure, $GD \perp CH$ and GD is the mean proportional between CD and DH. Prove $\triangle CDG \sim \triangle GDH$.

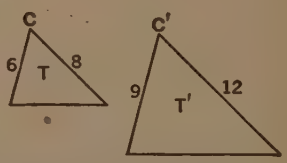


9. In the figure of Ex. 8, if $CG \perp GH$, prove rt. $\triangle CDG \sim$ rt. $\triangle CGH$.

(Supplementary Exercises 11–16, pp. 255, 256, may be studied now.)

Oral Exercises

1. In the annexed figure, $\angle C = \angle C'$. How do the triangles compare in respect to their areas? How do they compare in respect to their forms? Why?



2. Can superposition be used if two triangles (*a*) are mutually equiangular; (*b*) are mutually equilateral; (*c*) have sides which are respectively proportional?

3. If $a : b = c : d$ and $a : b = c : e$, then $d = e$. Why?

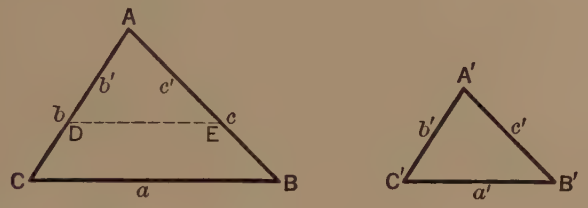
4. If two triangles are congruent are they also similar? Why?

5. May one of two congruent triangles be substituted for the other? Why?

6. $\triangle X \sim \triangle Z$ and $\triangle X \cong \triangle Y$. Explain why $\triangle Y \sim \triangle Z$. (Two ways.)

Proposition 3 (C N)

301. Theorem. *Triangles with proportional sides are similar.*



Given: $\triangle ABC$ and $A'B'C'$, $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

To prove: $\triangle ABC \sim \triangle A'B'C'$.

HINT: Form $\triangle ADE$ with $AD = b'$ and $AE = c'$. Prove $\triangle ADE \sim \triangle ABC$ and $\triangle ADE \cong \triangle A'B'C'$.

Proof: STATEMENTS	FACTS
1. Let $AD = b'$, $AE = c'$, and draw DE.	1. *Prins.* 10 and 13.
2. $\frac{b}{b'} = \frac{c}{c'}$.	2. Given.
3. $\angle A = \angle A$.	3. Identical.
4. $\triangle ABC \sim \triangle ADE$.	4. § 300.
5. $\frac{b}{b'} = \frac{a}{DE}$.	5. Def. of sim. polygons.
6. $\frac{b}{b'} = \frac{a}{a'}$.	6. Given.
7. $DE = a'$.	7. § 270.
8. From Statements 1 and 7, $\triangle ADE \cong \triangle A'B'C'$.	8. *s.s.s.* = *s.s.s.*
9. From Statements 4 and 8, $\triangle ABC \sim \triangle A'B'C'$.	9. § 298.

AERIAL PHOTOGRAPHY. (See p. 215.) *Courtesy International Newsreel*

Exercises. Similar Triangles

1. Two triangles are similar if two sides and the median to one of these sides of one triangle are proportional to the corresponding parts of the other triangle.

2. Two right triangles are similar if the hypotenuse and a leg of the one are proportional to the hypotenuse and the corresponding leg of the other.

HINT: Draw the median to the hypotenuse in each triangle.

302. Summary of Similar Triangles

Two triangles are similar if

a) two angles of the one are equal to two angles of the other;
b) an angle of the one equals an angle of the other and the including sides are proportional;
c) their sides are respectively proportional;
d) their sides are respectively parallel;
e) their sides are respectively perpendicular.

The importance of condition (*a*) has been emphasized as Tool XXIII.

303. From the definition of similar polygons (§ 292) and the proofs of Propositions 1 and 3, the following Tool may be stated.

Tool XXIV. (*a*) Corresponding angles of similar triangles are equal.
(*b*) Corresponding sides of similar triangles are proportional.
(*c*) Corresponding sides of similar triangles lie opposite equal angles.

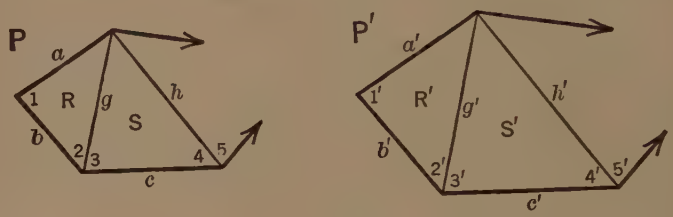
HOW AN AERIAL PHOTOGRAPH IS MADE

(See photograph, page 214.)

Before such a photograph is taken from an airplane, the line of flight, speed of the airplane, interval between exposures, and distance from the ground are calculated by the aid of geometry. Why are the successive pictures made to overlap? What triangles are congruent, similar, or equal? What lines are equal or parallel? Can you form a problem of your own based upon the lines in the picture?

Proposition 4 (C N)

304. Theorem. *Corresponding triangles of similar polygons are similar.*



Given: $P \sim P'$; g and g', h and h', etc., are corresponding diagonals making $\measuredangle R$ and R', $\measuredangle S$ and S', etc.

To prove: $\triangle R \sim \triangle R'$, $\triangle S \sim \triangle S'$, etc.

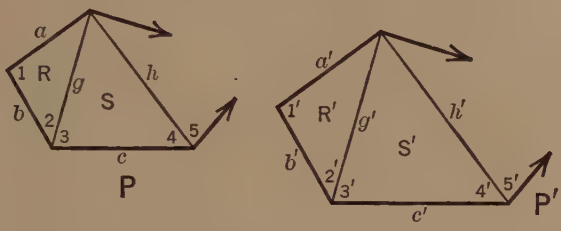
HINT: What facts are known about similar polygons? What is the usual Tool used to prove triangles similar?

Proof:

STATEMENTS	FACTS
1. $P \sim P'$.	1. Given.
2. $\frac{a}{a'} = \frac{b}{b'}$.	2. Def. of similar polygons.
3. $\angle 1 = \angle 1'$.	3. Why?
4. $\triangle R \sim \triangle R'$.	4. Why?
5. $\angle(2+3) = \angle(2'+3')$.	5. Def. of similar polygons.
6. $\angle 2 = \angle 2'$.	6. Why?
7. $\angle 3 = \angle 3'$.	7. Axiom of Subtraction.
In like manner it may be shown that $\angle 4 = \angle 4'$, $\angle 5 = \angle 5'$, etc.	
8. $\triangle S \sim \triangle S'$, etc.	8. *a.a.* = *a.a.*

Proposition 5 (c N)

305. Theorem. *Polygons formed by like-placed similar triangles are similar.*



Given: P and P'; $\triangle R \sim \triangle R'$, $\triangle S \sim \triangle S'$, etc.

To prove: $P \sim P'$.

HINT: What two conditions make polygons similar? What Facts are known about similar triangles?

Proof: STATEMENTS	FACTS
1. $\triangle R \sim \triangle R'$, $\triangle S \sim \triangle S'$, etc.	1. Given.
2. $\frac{a}{a'} = \frac{b}{b'} = \left(\frac{g}{g'}\right) = \frac{c}{c'} = \left(\frac{h}{h'}\right) =$ etc.	2. Corr. sides of similar triangles.
3. $\angle 1 = \angle 1'$, $\angle 2 = \angle 2'$, $\angle 3 = \angle 3'$, etc.	3. Why?
4. $\angle(2+3) = \angle(2'+3')$, $\angle(4+5) = \angle(4'+5')$, etc.	4. Why?
5. $P \sim P'$.	5. Definition.

Exercises. Similar Polygons

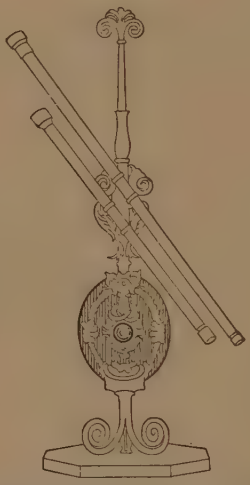
1. Within quadrilateral $ABCD$ join any point P to each of the vertices. From any point A' of PA draw a line parallel to AB intersecting PB in B'; from B' draw $B'C' \parallel BC$; continue in like manner and finally join D' and A'. Prove $ABCD \sim A'B'C'D'$.

2. Prove the above conclusion if P is taken as any point without $ABCD$.

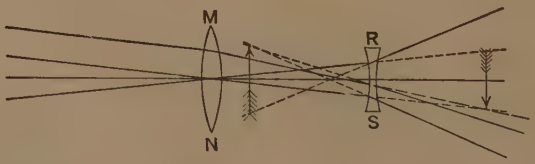
SIMILAR FIGURES IN ASTRONOMY

The construction of the telescope which is used in astronomy is dependent upon the relation of similar figures and the observations which are made by it are often interpreted by means of proportional parts of similar figures.

The earliest form of telescope was invented by Galileo, who made many important discoveries and observations in physics and astronomy.



Two of Galileo's telescopes are shown above. In the center of the oval ivory frame is the object glass, with which the four satellites of Jupiter were discovered in 1610. (The drawing was made from a photograph of these relics which are preserved in a museum in Italy.)



This diagram shows the principle of Galileo's telescope. *MN* is the object glass and *RS* the eyepiece. After you have studied the subject of Light or Optics in physics, you will be better able to appreciate the full meaning of the similar triangles shown.

PROPORTIONAL MAGNITUDES

Proposition 6

306. Theorem. *In a series of equal ratios the sum of the numerators and the sum of the denominators have the same ratio as any numerator and its denominator.*

Given: $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$.

To prove: $\frac{a+c+e}{b+d+f} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$.

Proof:

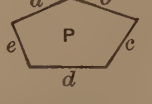
STATEMENTS	FACTS
1. Let the value of each ratio be represented by r; then $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = r$.	1. Given.
2. $a = br$. $c = dr$. $e = fr$.	2. Axiom of Multiplication.
3. $a + c + e = r(b + d + f)$.	3. Why?
4. $\frac{a+c+e}{b+d+f} = r$.	4. Why?
5. $\frac{a+c+e}{b+d+f} = (r) = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$.	5. Sub.

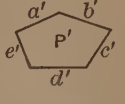
Oral Exercises

1. Apply the preceding theorem to the series $\frac{2}{4} = \frac{3}{6} = \frac{5}{10} = \frac{7}{14}$.

2. Write a series of equal ratios beginning with $\frac{2}{3}$ and apply § 306.

3. If P and P' are similar pentagons, write a series of equal ratios of their sides.





4. In the preceding exercise, what would be represented by the sum of the five numerators? by the sum of the five denominators?

Proposition 7 (c N)

307. Theorem. *The perimeters of similar polygons have the same ratio as their corresponding sides.*

Given: P and P' with perimeters p and p'; corresponding sides a and a', etc.

To prove: $\frac{p}{p'} = \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$, etc.

HINT: What series of equal ratios can be obtained from two similar polygons?

Proof:

STATEMENTS	FACTS
1. $P \sim P'$.	1. Given.
2. $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'}$, etc.	2. § 292.
3. $p = a + b + c + d$, etc. $p' = a' + b' + c' + d'$, etc.	3. *Prin.* 4.
4. $\frac{a + b + c + d, \text{etc.}}{a' + b' + c' + d', \text{etc.}} = \frac{a}{a'}$.	4. Why?
5. $\frac{p}{p'} = \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$, etc.	5. Sub.

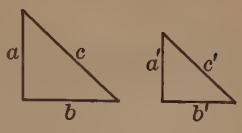
Sight Exercises

1. If two corresponding sides of two similar polygons are 15 and 20, what is the ratio of their perimeters?

2. In Ex. 1, if the perimeter of the larger polygon is 60, what is the perimeter of the smaller polygon?

3. In Ex. 1, if the perimeter of the smaller polygon is 60, what is the perimeter of the larger polygon?

4. If $\triangle abc$ and $\triangle a'b'c'$ are isosceles right triangles with $c = 24$ and $c' = 16$, what is the ratio (a) of their perimeters? (b) of b to b'?



5. $\triangle T \sim \triangle T'$. Find the perimeter of $\triangle T$, if its sides are a, b, 9 and the sides of $\triangle T'$ are 8, 5, 6 respectively.

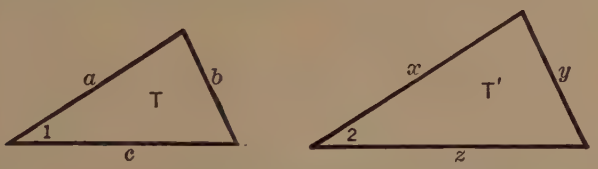
6. If the perimeter of an equilateral triangle is doubled are all of its sides doubled? Is each angle doubled?

7. If $x = 4$, why does $x^2 = 16$? (*Prin.* 9.)

8. If $\frac{a}{x} = \frac{b}{y}$, why does $\frac{a^2}{x^2} = \frac{b^2}{y^2}$?

Proposition 8 (C N)

308. Theorem. *Similar triangles have the same ratio as the squares of their corresponding sides.*



Given: $\triangle T \sim \triangle T'$ with corresponding sides a and x, etc.

To prove: $\frac{\triangle T}{\triangle T'} = \frac{a^2}{x^2} = \frac{b^2}{y^2} = \frac{c^2}{z^2}$.

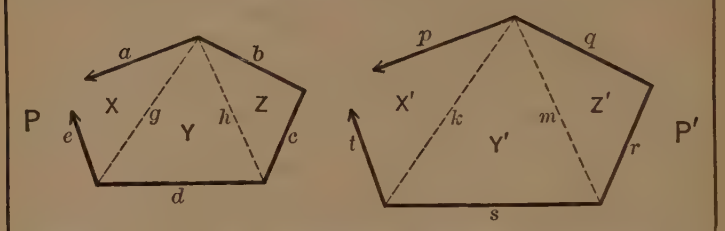
HINT: What Fact relates to the ratio of similar polygons?

Proof:

STATEMENTS	FACTS
1. $\triangle T \sim \triangle T'$.	1. Given.
2. $\angle 1 = \angle 2$.	2. Why?
3. $\frac{\triangle T}{\triangle T'} = \frac{ac}{xz} = \frac{a}{x} \cdot \frac{c}{z}$.	3. Why?
4. $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$.	4. Corr. sides of similar triangles.
5. $\frac{a^2}{x^2} = \frac{b^2}{y^2} = \frac{c^2}{z^2}$.	5. Why?
6. $\frac{\triangle T}{\triangle T'} = \frac{a}{x} \cdot \frac{a}{x} = \frac{a^2}{x^2} = \frac{b^2}{y^2} = \frac{c^2}{z^2}$.	6. Sub.

Proposition 9 (C N)

309. Theorem. *Similar polygons have the same ratio as the squares of their corresponding sides.*



Given: $P \sim P'$ with corresponding sides a and p, etc.

To prove: $\dfrac{P}{P'} = \dfrac{a^2}{p^2} = \dfrac{b^2}{q^2}$, etc.

HINT: How are similar triangles related in reference to a pair of corresponding sides?

Proof: STATEMENTS	FACTS
1. Draw the corresponding diagonals g, h, k, and m dividing the similar polygons into corresponding triangles.	1. Given and § 290.
2. $\triangle X \sim \triangle X'$, $\triangle Y \sim \triangle Y'$, $\triangle Z \sim \triangle Z'$.	2. § 304.
3. $\dfrac{\triangle X}{\triangle X'} = \dfrac{g^2}{k^2}$, $\dfrac{\triangle Y}{\triangle Y'} = \dfrac{g^2}{k^2} = \dfrac{h^2}{m^2}$, $\dfrac{\triangle Z}{\triangle Z'} = \dfrac{h^2}{m^2}$.	3. § 308.
4. $\dfrac{\triangle X}{\triangle X'} = \left(\dfrac{g^2}{k^2}\right) = \dfrac{\triangle Y}{\triangle Y'} = \left(\dfrac{h^2}{m^2}\right) = \dfrac{\triangle Z}{\triangle Z'}$.	4. Sub.
5. $\dfrac{\triangle X + \triangle Y + \triangle Z}{\triangle X' + \triangle Y' + \triangle Z'} = \dfrac{\triangle X}{\triangle X'}$.	5. § 306.
6. $\dfrac{P}{P'} = \dfrac{\triangle X}{\triangle X'}$.	6. *Prin.* 4 and Sub.

7. $\frac{\triangle X}{\triangle X'} = \frac{a^2}{p^2}$.	7. § 308
8. $\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$, etc.	8. § 292.
9. $\frac{a^2}{p^2} = \frac{b^2}{q^2} = \frac{c^2}{r^2}$, etc.	9. *Prin.* 9.
10. $\frac{P}{P'} = \frac{\triangle X}{\triangle X'} = \frac{a^2}{p^2} = \frac{b^2}{q^2}$, etc.	10. Sub.

Oral Exercises

1. If two corresponding sides of two similar polygons are 6 and 12, what is the ratio of their areas? of their perimeters?

2. In Ex. 1, if the area of the smaller polygon is 36, what is the area of the larger polygon?

3. In Ex. 1, if the area of the larger polygon is 36, what is the area of the smaller polygon?

4. How many times would the area of an equilateral triangle be increased (*a*) if its sides were doubled? (*b*) if its sides were tripled?

Sight Exercises. Perimeters and Areas of Similar Polygons

1. If $x^2 = 25$, why does $x = 5$? (*Prin.* 9.)

2. If $\frac{a^2}{b^2} = \frac{9}{16}$, then $\frac{a}{b} = \frac{3}{4}$. Why?

3. If $\frac{x^2}{4} = \frac{9}{16}$, show that $x = 1\frac{1}{2}$ by using *Prin.* 9.

4. The areas of two similar polygons are respectively 49 and 64 square units. If a side of the first polygon is 9, find the corresponding side of the second.

HINT: $\frac{49}{64} = \frac{9^2}{x^2}$. Why?

5. The areas of two similar polygons are respectively 81 and 144. If a side of the smaller polygon is 16, find the corresponding side of the larger polygon.

6. If two polygons are similar and one is five times as large as the other, find the ratio of their corresponding sides.

HINT: $\frac{P}{P'} = \frac{5}{1} = \frac{a^2}{b^2}$. Why?

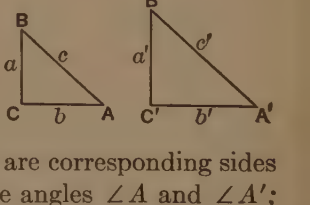
7. What is the ratio of two corresponding sides of two similar polygons, if the ratio of the areas is (*a*) 4 : 1? (*b*) 2 : 1? (*c*) 25 : 16? (*d*) $2\frac{1}{4}$: 1?

8. If the ratio of the perimeters of two similar polygons is 4 : 1, what is the ratio of their areas?

9. If the ratio of the areas of two similar polygons is 4 : 1, what is the ratio of their perimeters?

10. Compose an exercise of your own about perimeters and areas of two similar polygons with two corresponding sides known.

310. The Ratios of Corresponding Sides of Similar Triangles. In a number of theorems which follow, it will be required to write the ratios of the corresponding sides of similar triangles. If the pairs of corresponding sides are not known, they can be located by Part *c* of Tool XXIV (p. 215), which says, *Corresponding sides of similar triangles lie opposite equal angles.*



In the annexed figure, if $\angle C = \angle C' = \text{rt. } \angle$, and $\angle A = \angle A'$, then $\triangle ABC \sim \triangle A'B'C'$ (Why?) and $\angle B = \angle B'$. (Why?) c and c' are corresponding sides since they lie opposite the right angles; a and a' are corresponding sides since they lie opposite the equal acute angles $\angle A$ and $\angle A'$; b and b' are corresponding sides since they lie opposite the equal acute angles $\angle B$ and $\angle B'$. Since by Part *b* of Tool XXIV, *Corresponding sides of similar triangles are proportional*, we may write $\frac{c}{c'} = \frac{a}{a'} = \frac{b}{b'}$.

It should be noted that, all *the numerators are obtained from one triangle* ($\triangle ABC$), and the denominators from the other triangle ($\triangle A'B'C'$), and that *any denominator is the corresponding part to its numerator.*

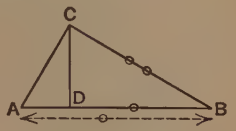
If it is desired to prove four lines proportional, try to find a pair of triangles each having two of the terms of the proportion as sides. If these triangles can be proved similar,

their corresponding sides will be proportional. Marking the sides which are terms of the desired proportion will assist in determining what triangles are to be proved similar.

Illustrative Example:

Given: rt. $\triangle ABC$ with $CD \perp AB$.

To prove: $\frac{AB}{BC} = \frac{BC}{DB}$.



Mark these four lines as shown in the figure. Since AB can be associated only with BC as sides of $\triangle ACB$ and the remaining two lines, BC and BD, as sides of $\triangle CDB$, we will try to prove $\triangle ACB \sim \triangle CDB$.

$\triangle ACB$ and $\triangle CDB$ are right triangles. Given.
$\angle CBD = \angle CBA$ by identity.
$\triangle ACB \sim \triangle CDB$. (§ 295).

In writing the proportion of the corresponding sides, the *first term of the required conclusion* (in this case, AB) may be called the *key term of the proportion*—in that it tells us the *order* in which the remaining three terms *must be written.*

Because AB (the key term) lies opposite the right angle C, then the denominator of AB must lie opposite the equal of $\angle C$, that is $\angle D$, and we have the first ratio $\frac{AB}{BC}$.

Since AB (the key term) is part of $\triangle ACB$, then the numerator of the second ratio must come from the same triangle, and we have $\frac{AB}{BC} = \frac{BC}{?}$.

The only line left is DB, and the complete proportion is $\frac{AB}{BC} = \frac{BC}{DB}$.

(It should be noted that DB of $\triangle CDB$ is the corresponding side to BC of $\triangle ACB$.)

THE 100-INCH HOOKER TELESCOPE AT MT. WILSON OBSERVATORY

This reflector telescope has added half a billion stars to human ken. The movable parts weigh 100 tons, but the great instrument can be shifted and its position determined down to infinitesimally small fractions of an inch. The mirror that gathers the starlight and sends it to the eyepiece is at the lower end of the big structural steel barrel. The eyepiece is at the side, and may be placed near the top or bottom, depending on the type of work being done. The size of this instrument may be better appreciated by comparing it with that of a chair shown, which may be located to the left and below the center of the picture and on the far side of the telescope.

311. Method of Proving Four Lines Proportional. The usual means of proving four lines proportional is,

(1) try to find a pair of triangles each having two of these lines as sides; (It is advisable to mark the lines as indicated in the previous section.)

(2) prove the triangles similar;

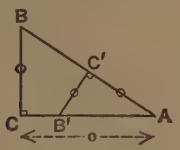
(3) state a proportion of the marked sides;

(4) if necessary, transform the proportion by alternation.

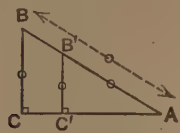
312. Method of Proving Two Products Equal. The usual means of proving the product of two lines equal to the product of two other lines is to prove the four lines proportional as in the previous method and take the product of the means equal to the product of the extremes.

Sight Exercises. Proportional Lines

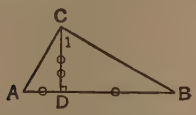
In the following exercises, 1–6, write a proportion of the marked sides by using the method described in § 310.



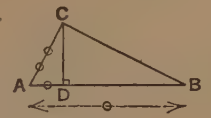
1. $\angle C = \angle C' = 90°$.



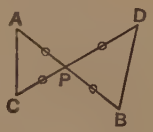
2. $\angle C = \angle C' = 90°$.



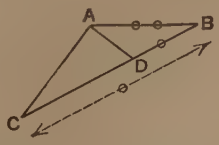
3. $CD \perp AB$ and $\angle A = \angle 1$.



4. $CD \perp AB$.



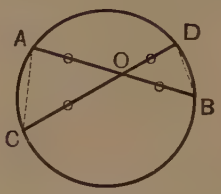
5. $\angle A = \angle D$.



6. $\angle BAC = \angle ADB$.

Proposition 10 (C N)

313. Theorem. *The product of the segments of one of two intersecting chords equals the product of the segments of the other.*



Given: AB and CD intersect within the circle at any point O.

To prove: $AO \times BO = CO \times DO$.

HINT: What directions have been given for proving two products equal? (See § 312.)

Proof:

STATEMENTS	FACTS
1. Draw AC and BD. In $\triangle AOC$ and $\triangle BOD$,	1. *Prin.* 13.
2. $\angle A$ = numerically $\frac{1}{2}\overset{\frown}{BC}$, $\angle D$ = numerically $\frac{1}{2}\overset{\frown}{BC}$.	2. Why?
3. $\angle A = \angle D$. In like manner $\angle C = \angle B$.	3. Why?
4. $\triangle AOC \sim \triangle BOD$.	4. Why?
5. $\frac{AO}{DO} = \frac{CO}{BO}$.	5. Why?
6. $AO \times BO = CO \times DO$.	6. Why?

Question. Could we have proved the theorem if we had drawn AD and CB?

NOTE: Proposition 10 is sometimes stated: *If two chords intersect within a circle, the product of their segments is a constant.* This means that the product of the segments of any chord through O has a fixed value.

Exercises. Intersecting Chords

1. In the figure, three segments of the intersecting chords are known. Find the fourth segment.

	p	r	s	t
a	?	3	6	4
b	21	4	?	7
c	12	4	8	?

2. Find the segments of chord PR, if its length and the segments of the chord which intersect it are known.

	PR	p	r
a	7	6	2
b	9	2	9
c	10	3	8

3. If chord PR is bisected by ST, find the segments of PR.

	p	r
a	4	9
b	16	25

4. If ST is a diameter $\perp PR$, find the segments of PR.

	p	r
a	4	16
b	9	25

5. If $p = 10$ and $r = 8$, what is the product of the segments of any other chord through the point of intersection? Does this illustrate that *the product of the segments is a constant?*

6. In the annexed figure, prove $\frac{p}{t} = \frac{s}{r}$.

7. The radius of a circle is 15 in. and a point P is 12 in. from the center. Find the lengths of the longest and the shortest chords through point P.

8. Find the length of a chord if the mid-points of its arcs are 2 in. and 18 in. respectively from the chord.

9. The arch of a bridge is an arc of a circle with a span of 48 ft. and a height of 18 ft. Find the radius of the arc.

10. The mid-points of two chords are 5 and 9 in. respectively from the mid-points of their arcs. The length of the first chord is 20 in. Find the length of the second chord.

(Supplementary Exercises 17–22, p. 256, may be studied now.)

Oral Exercises

1. What is the name of a proportion such as $a : b = b : c$? Why is it so called? What is the name of the term b?

2. Why does $b^2 = ac$ in the proportion of Ex. 1?

3. If you are required to prove that a line is the mean proportional between two other lines, what line must be marked twice in the figure?

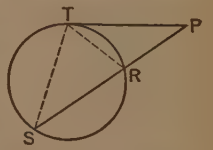
4. A mean proportional is generally written as which two terms of a proportion?

5. What is the mean proportional between 25 and 9? x^2 and y^2? g and h?

6. If you are to use similar triangles in proving $OC : OA = OA : OB$, name the two triangles.

7. State the theorems concerning an inscribed angle and an angle formed by a tangent and a chord. Under what condition are two such angles equal?

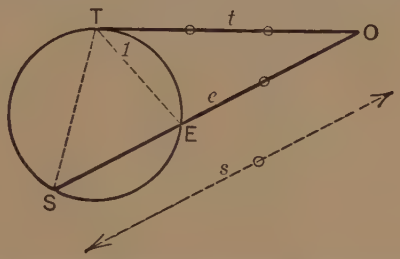
8. In the annexed figure, if you are required to prove $\triangle PTR \sim \triangle PST$, what angles would you try to prove equal?



NOTE: When proving angles equal, it is helpful in many cases to select angles which are of a special kind, such as, inscribed angles, angles formed by a tangent and a chord, right angles, common angles, etc., and try to find a relation between them.

Proposition 11 (c n)

314. Theorem. *A tangent from a point to a circle is the mean proportional between a secant from that point and its external segment.*



Given: t is a tangent and s a secant from the same external point (O) to the circle; e is the external segment of the secant.

To prove: $s : t = t : e$.

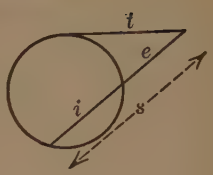
HINT: What directions have been given for proving four lines proportional? (See § 311.)

Proof: STATEMENTS	FACTS
1. Draw ST and ET. In $\triangle OST$ and $\triangle OTE$,	1. By what Principle?
2. $\angle O = \angle O$.	2. Why?
3. $\angle S$ = numerically $\frac{1}{2}\ \overset{\frown}{TE}$, and $\angle 1$ = numerically $\frac{1}{2}\ \overset{\frown}{TE}$.	3. Why?
4. $\angle S = \angle 1$.	4. Why?
5. $\triangle OST \sim \triangle OTE$.	5. *a.a.* = *a.a.*
6. $s : t = t : e$.	6. Why?

Questions. 1. Why are TE and TS drawn? 2. If $\angle TES = 90°$, what special line is TS? What special angle is STO?

Exercises. Tangents and Secants

In the following exercises, 1–4, t is a tangent, e is the external segment, and i is the internal segment of secant s.



1. Find the length of t when s and e are known for each set of values.

	s	e	t
a	9	4	?
b	25	4	?
c	25	9	?

2. Find e for each set of values in the following:

	t	s	e
a	9	27	?
b	8	16	?
c	10	20	?

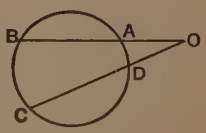
3. From the internal segment i and the external segment e, find the tangent in each of the following:

	i	e	t
a	5	4	?
b	16	9	?
c	12	4	?

4. Find s in the following:

	i	t	s
a	9	6	?
b	7	12	?

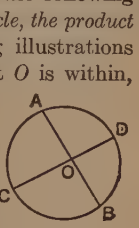
5. In the annexed figure, secants OB and OC are drawn from a point without a circle. Prove $\frac{OB}{OC} = \frac{OD}{OA}$.



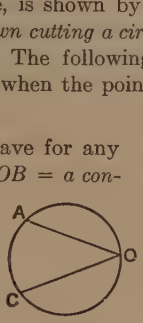
6. In the figure of Ex. 5, prove $OB \times OA = OC \times OD$.

7. In the figure of Ex. 5, prove $\triangle OAD \sim \triangle OCB$.

315. Example of the Principle of Continuity. The continuity of Propositions 10, 11, and Exercise 6, above, is shown by the following statement: *If through a point a line is drawn cutting a circle, the product of the segments of the line is a constant.* The following illustrations show that this statement may be applied when the point O is within, on, or without the circle:

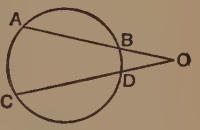


If the point O is within the circle, we have for any position of chord CD, $OC \times OD = OA \times OB =$ *a constant.*

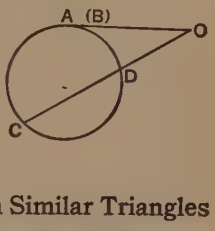


If O moves to a point on the circle, OB = zero and OD = zero and we have for any position of chord OC, $OC \times OD$ = zero = *a constant.*

If O moves without the circle, we have for any position of secant ODC, $OC \times OD = OA \times OB =$ *a constant.*



If O so moves that B approaches A and finally falls upon A, then $OB = OA$ and we have for any position of secant ODC, $OC \times OD = (OA \times OB) = (OA)^2 =$ *a constant.*



What is the constant value of $OC \times OD$ when O is at the center of the circle?

Exercises. Proportions and Products from Similar Triangles

1. How high is a tree that casts a shadow 50 ft. long, if an 8-foot vertical post casts a shadow 10 ft. long at the same time?

2. In $\triangle ABC$ altitudes AH and BG intersect at O. Prove $OA \times BH = OB \times AG$.

3. Use the conditions of Ex. 2 to prove $AC : BC = CH : CG$.

4. Corresponding altitudes of similar triangles have the same ratio as any two corresponding sides.

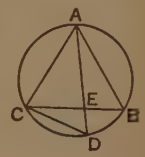
5. The diagonals of a trapezoid whose bases are AB and CD intersect at O. If $BO = 9$, $DO = 3$, and $AC = 16$, find OA and OC.

6. In the annexed figure, if $\angle ADB = \angle ACE$, prove $AB \times AC = AD \times AE$.

7. Two triangles are similar and the sides of the one are 6, 8, and 12. If the shortest side of the other is 10, find the remaining sides.

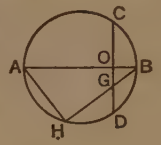
8. Corresponding medians of similar triangles have the same ratio as any two corresponding sides.

9. BC is the base of an isosceles $\triangle ABC$ inscribed in a circle. If chord AD cuts BC at E, prove $AD : AB = AB : AE$.



10. In the figure of Ex. 9, prove $AD : AC = AC : AE$.

11. In the annexed figure, AB is a diameter perpendicular to CD. Prove $AB \times OB = BH \times BG$.



12. In the figure of Ex. 11, prove $OB : OC = OD : OA$.

13. In the annexed figure, if E is a point in chord AB, and EC is perpendicular to the diameter AD, prove $AC \times AD = AB \times AE$.

14. Prove by a proportion that a line joining the mid-points of two sides of a triangle equals half the third side.

HINT: $\triangle ADE \sim \triangle ABC$; $\frac{AE}{AC} = \frac{1}{2}$.

15. The diagonals of a trapezoid divide each other proportionally.

16. The sides of a triangle are 9 in., 12 in., and 15 in. respectively. A line parallel to the 15 in. base is 10 in. long. Find the segments into which it divides the other two sides.

17. Two circles intersect at A and B. AC and AD are each a chord of one circle and a tangent to the other circle. Prove $BC : AB = AB : BD$.

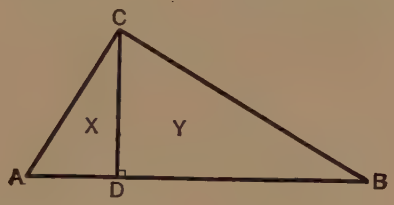
18. In the annexed figure, if $\overset{\frown}{AC} = \overset{\frown}{BC}$, prove $CE : AC = AC : CD$.

19. In similar triangles the radii of the circumscribed circles have the same ratio as two corresponding sides.

(Supplementary Exercises 23–30, p. 257, may be studied now.)

Proposition 12 (c N)

316. Theorem. *The altitude upon the hypotenuse of a right triangle forms two triangles each similar to the given triangle and to the other.*



Given: $\triangle ABC$, C is a rt. $\angle$, $CD \perp AB$.

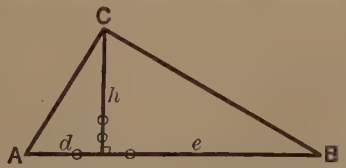
To prove: rt. $\triangle X \sim$ rt. $\triangle ACB \sim$ rt. $\triangle Y$.

HINT: What condition makes right triangles similar?

Proof:

STATEMENTS	FACTS
1. $\triangle ACB$, $\triangle X$, and $\triangle Y$ are rt. $\triangle$.	1. Given.
2. $\angle A = \angle A$.	2. Iden.
3. rt. $\triangle X \sim$ rt. $\triangle ACB$.	3. § 295.
4. $\angle B = \angle B$.	4. Iden.
5. rt. $\triangle Y \sim$ rt. $\triangle ACB$.	5. Why?
6. rt. $\triangle X \sim$ rt. $\triangle Y$.	6. § 298.

317. Corollary 1. *The altitude upon the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse.*

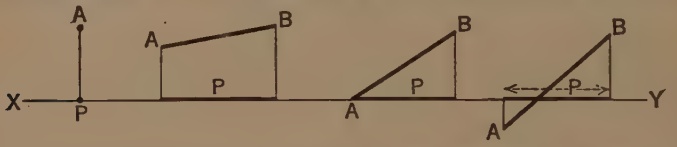


HINT: What is a mean proportional? What directions have been given for proving four lines proportional?

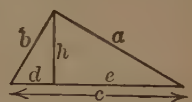
Proof:

STATEMENTS	FACTS
1. rt. $\triangle X \sim$ rt. $\triangle Y$.	1. § 316.
2. $\frac{d}{h} = \frac{h}{e}$	2. § 292.

318. Projections. The projection of a point upon a line is the *foot* of the perpendicular from the point to the line. The projection of one line upon another is the *segment between the projections of the extremities of the one line upon the other.*



In the figures above, point P is the projection of point A upon the line XY and the segment P in each of the other figures is the projection of the line AB upon the line XY.

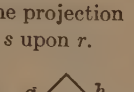


In the annexed figure, c is the hypotenuse of rt. $\triangle abc$ and $h \perp c$. The projection of leg b upon the hypotenuse is d and the projection of leg a upon the hypotenuse is e.

Oral Exercises

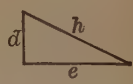
1. In the obtuse triangle prs, sketch the projection of p upon s; of r upon s; of s upon p; of s upon r.

2. In the isosceles triangle ghk, what two projections are equal?



3. In an equilateral triangle, how many projections are equal?

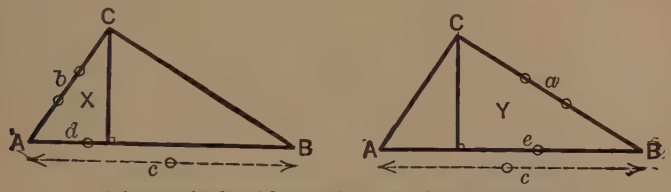
4. In rt. $\triangle deh$, what is the projection of d upon e? of e upon d? of h upon e? of h upon d?



5. The altitude upon the hypotenuse of a right triangle is the mean proportional between what two projections?

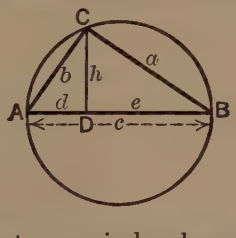
6. In an acute triangle, what does the sum of the projections of any two sides upon the third equal?

319. Corollary 2. *Either leg of a right triangle is the mean proportional between the hypotenuse and its projection upon the hypotenuse.*



(The pupil should complete the demonstration.)

320. Corollary 3. *The perpendicular from any point on a circle to a diameter is the mean porportional between the segments of the diameter; and, the chord joining the point to either extremity of the diameter is the mean proportional between the diameter and its projection upon the diameter.*



Given: C is any point on a circle whose diameter is AB and $CD \perp AB$.

To prove: (1) $d : h = h : e$,
(2) $c : b = b : d$,
(3) $c : a = a : e$.

HINT: What kind of an angle is $\angle C$? Why? Use §§ 317 and 319.

(The pupil should complete the demonstration.)

Exercises. Right Triangles

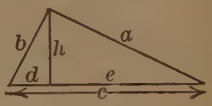
1. If c is the hypotenuse of right triangle abc and h is the altitude upon the hypotenuse, what three lines are mean proportionals? Write the three mean proportions.

2. The hypotenuse of a right triangle is 75 and one leg is 60. Find the altitude upon the hypotenuse.

3. The segments of an hypotenuse formed by the altitude upon it are 45 and 80. Find the three sides of the triangle.

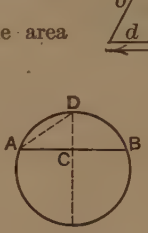
4. If one leg of a right triangle is 60 and its projection upon the hypotenuse is 36, what are the other sides of the triangle?

5. If $e = 112$ and $h = 84$, find the three sides of the triangle.

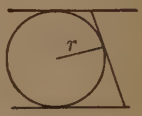


6. If $b = 135$ and $d = 81$, find the area of rt. $\triangle$ abc. (Two methods.)

7. C and D are the mid-points of a chord AB and of its arc. If $AD = 12$ and $CD = 8$, find the radius of the circle.

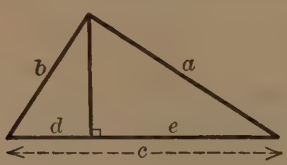


8. Between two parallel tangents a third tangent is drawn. Show that the radius is a mean proportional between the segments of the third tangent.



9. If one leg of a right triangle is double the other, the perpendicular from the vertex of the right angle to the hypotenuse divides the hypotenuse into segments which have the ratio of 4 : 1.

321. Corollary 4 (C). *The square of the hypotenuse of a right triangle equals the sum of the squares of the legs.*



Given: c is the hypotenuse, a and b are legs of right triangle abc.

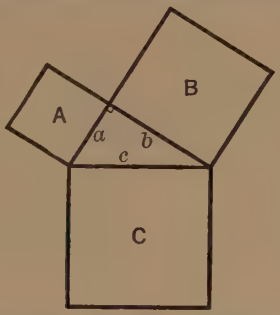
To prove: $c^2 = a^2 + b^2$.

HINT: Use the Substitution Method, § 103.

Proof:

STATEMENTS	FACTS
1. $\frac{c}{a} = \frac{a}{e}$ and $\frac{c}{b} = \frac{b}{d}$.	1. § 319.
2. $a^2 = ce$ and $b^2 = cd$.	2. Why?
3. $a^2 + b^2 = ce + cd$ $= c(e + d)$.	3. Axiom of Addition.
4. $(e + d) = c$.	4. Why?
5. $a^2 + b^2 = c \cdot c = c^2$.	5. Sub.

322. Corollary 5 (N). *The square on the hypotenuse of a right triangle equals the sum of the squares on the legs.*



Given: C is the square on the hypotenuse and A and B are the squares on the legs of right triangle *abc*.

To prove: $C = A + B$.

HINT: What are the dimensions of each of the squares? Rectangles have what ratio?

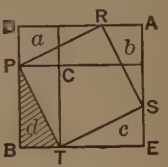
Proof:

STATEMENTS	FACTS
1. $\frac{A}{C} = \frac{a^2}{c^2}$ and $\frac{B}{C} = \frac{b^2}{c^2}$.	1. § 237.
2. $\frac{A + B}{C} = \frac{a^2 + b^2}{c^2}$.	2. Axiom of Addition.
3. $a^2 + b^2 = c^2$.	3. § 321.
4. $\frac{A + B}{C} = \frac{c^2}{c^2} = 1$.	4. Sub.
5. $A + B = C$.	5. *Prin.* 2.

323. The Pythagorean Theorem. Since the square *on a line* is measured by the square *of the line*, the proofs of Corollaries 4 and 5 are of the same fact. The conclusion of the first is *algebraic* while that of the second is *geometric*.

This theorem is called *The Pythagorean Theorem* because it is supposed that Pythagoras (see p. 384) was the first to prove it. The truth of the theorem was known, without doubt, long before his time to the Egyptians and the Hindus. The theorem is in many respects the most famous and the most useful fact of all geometry. Its proof has attracted the attention of a great many mathematicians.

A few proofs other than those given in §§ 321 and 322 are suggested in the following:



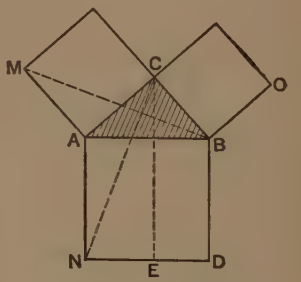
(1) Rectangle AC = rectangle CB.

$\triangle a = \triangle b = \triangle c = \triangle d = \frac{1}{2} AC = \frac{1}{2} CB$.

$AB - (\triangle a + \triangle b + \triangle c + \triangle d) = AB - (AC + BC)$.

$PRST = CD + CE$.

The proof suggested here is attributed to Pythagoras.



(2) $MC = 2 \triangle BAM$.

$AE = 2 \triangle CAN$.

$\triangle BAM \cong \triangle CAN$.

$MC = AE$.

$CO = BE$.

$MC + CO = AE + BE = AD$.

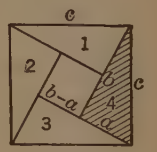
This proof is due to Euclid (p. 385).

(3) $\triangle 1 = \triangle 2 = \triangle 3 = \triangle 4$.

$$c^2 = 4\frac{ab}{2} + (b-a)^2.$$

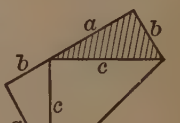
$$= 2ab + b^2 - 2ab + a^2.$$

$$= a^2 + b^2.$$



Bhaskara, a Hindu, used this proof.

(4) President James A. Garfield proved the theorem by showing that the annexed figure is a trapezoid and



$$\frac{(a+b)(a+b)}{2} = \frac{c^2}{2} + 2\frac{ab}{2}.$$

$$(a+b)(a+b) = c^2 + 2ab.$$

$$a^2 + 2ab + b^2 = c^2 + 2ab.$$

$$a^2 + b^2 = c^2.$$

SPECIAL FORMULAS

324. Tool XXV. By the Pythagorean Theorem, the value of any side of a right triangle may be found if the other two sides are known. Use the formula, $a^2 + b^2 = c^2$. (c is the hypotenuse.)

Exercises. The Pythagorean Theorem

1. Find the hypotenuse of a right triangle whose legs are 4.9 and 16.8.

Solution:

$a^2 + b^2 = c^2.$
$(4.9)^2 + (16.8)^2 = c^2.$
$24.01 + 282.24 = c^2.$
$306.25 = c^2.$
$17.5 = c.$

1 7. 5
3 06. 25
$1^2 = 1$
$2 \times 1 = 2$ | 206
27 | 189 $\quad 189 = 27 \times 7.$
$2 \times 17 = 34$ | 1725
345 | 1725 $\quad 1725 = 345 \times 5.$

2. In right triangle abc, find c if a and b are 18 and 24; 15 and 36; 14 and 48. (A table of square roots follows.)

3. Find the base of a right triangle whose altitude is 35 and whose hypotenuse is 91.

Solution:

$a^2 + b^2 = c^2.$
$(35)^2 + b^2 = (91)^2.$
$1225 + b^2 = 8281.$
$b^2 = 7056.$
$b = 84.$

4. Find the base of a right triangle if a and c are 21 and 35.

5. Find the altitude of a right triangle if b and c are 72 and 75.

6. Find the side of a square whose diagonal is 12.

7. Find the diagonals of a rectangle whose dimensions are 8 by 15; 11 by 60; 13 by 84.

8. A ladder is 25 ft. long and its base is placed 7 ft. from a wall. How high does the ladder reach?

9. How far is a pedestrian from his starting point if he walks 5 mi. N. and then 12 mi. E.? 10 mi. N., then 4 mi. E. and then 5 mi. N.?

10. A tree was broken 35 ft. from the ground and the top in falling struck a spot 12 ft. from the trunk. What was the height of the tree?

11. A train moving 30 mi. an hour on an elevated track crosses a trolley car traveling at right angles to it at 15 mi. an hour. How far apart will they be in 20 minutes?

12. A chord 24 in. long is 5 in. from the center of a circle. How far from the center is a chord 10 in. long?

13. A rectangle is inscribed in a circle with diameter 13. If one side of the rectangle is 5, what are the other sides?

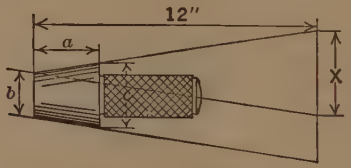
14. A pole 25 ft. long rests against a wall at a point 20 ft. from the ground. (*a*) If the top slides down 1 ft. along the wall, how far does the foot of the pole move from the wall? (*b*) If the foot moves out 1 ft. how far does the top descend?

15. A pirate buried a treasure 125 ft. from an old oak tree and in a direction N.E. from the tree. A searching party measured in a direction S.E. from the tree and 125 ft. out. How far were they from the true position and in what direction from it?

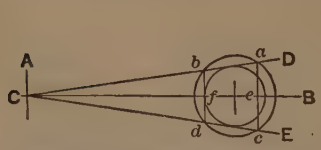
16. The centers of two circles are 25 in. apart. Find the lengths of their common internal tangents if the radii are 4 in. and 3 in.?

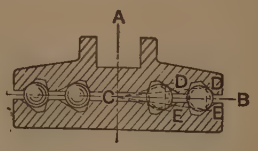
17. The radii of two circles are 5 and 12 and the distance between their centers is 25. Find the length of their common external tangent.

(Supplementary Exercises 31–42, p. 257, may be studied now.)



The formula for the taper per foot is $x = \frac{12(c-b)}{a}$. Why? (The difference of the diameters of the ends of the piece of material is called its taper.)





The laying out of circular ball-races, their position, the calculation of friction between parts, etc., depend upon the proportional parts of similar triangles.

SIMILAR TRIANGLES IN MACHINE PARTS

Squares and Square Roots

No.	Squares	Square Roots	No.	Squares	Square Roots	No.	Squares	Square Roots
1	1	1.000	34	1,156	5.831	67	4,489	8.185
2	4	1.414	35	1,225	5.916	68	4,624	8.246
3	9	1.732	36	1,296	6.000	69	4,761	8.307
4	16	2.000	37	1,369	6.083	70	4,900	8.367
5	25	2.236	38	1,444	6.164	71	5,041	8.426
6	36	2.449	39	1,521	6.245	72	5,184	8.485
7	49	2.646	40	1,600	6.325	73	5,329	8.544
8	64	2.828	41	1,681	6.403	74	5,476	8.602
9	81	3.000	42	1,764	6.481	75	5,625	8.660
10	100	3.162	43	1,849	6.557	76	5,776	8.718
11	121	3.317	44	1,936	6.633	77	5,929	8.775
12	144	3.464	45	2,025	6.708	78	6,084	8.832
13	169	3.606	46	2,116	6.782	79	6,241	8.888
14	196	3.742	47	2,209	6.856	80	6,400	8.944
15	225	3.873	48	2,304	6.928	81	6,561	9.000
16	256	4.000	49	2,401	7.000	82	6,724	9.055
17	289	4.123	50	2,500	7.071	83	6,889	9.110
18	324	4.243	51	2,601	7.141	84	7,056	9.165
19	361	4.359	52	2,704	7.211	85	7,225	9.220
20	400	4.472	53	2,809	7.280	86	7,396	9.274
21	441	4.583	54	2,916	7.348	87	7,569	9.327
22	484	4.690	55	3,025	7.416	88	7,744	9.381
23	529	4.796	56	3,136	7.483	89	7,921	9.434
24	576	4.899	57	3,249	7.550	90	8,100	9.487
25	625	5.000	58	3,364	7.616	91	8,281	9.539
26	676	5.099	59	3,481	7.681	92	8,464	9.592
27	729	5.196	60	3,600	7.746	93	8,649	9.644
28	784	5.292	61	3,721	7.810	94	8,836	9.695
29	841	5.385	62	3,844	7.874	95	9,025	9.747
30	900	5.477	63	3,969	7.937	96	9,216	9.798
31	961	5.568	64	4,096	8.000	97	9,409	9.849
32	1,024	5.657	65	4,225	8.062	98	9,604	9.899
33	1,089	5.745	66	4,356	8.124	99	9,801	9.950

325. Pythagorean Numbers. The following groups of three integers may represent the sides of a right triangle: 3, 4, 5; 5, 12, 13; 6, 8, 10; 7, 24, 25; 8, 15, 17; 9, 12, 15. Such groups are called *Pythagorean Numbers*. Many others may be added to this list by the use of the following formulas:

(1) $n, \frac{n^2-4}{4}, \frac{n^2+4}{4}$ when n is even;

(2) $n, \frac{n^2-1}{2}, \frac{n^2+1}{2}$ when n is odd.

Oral Exercises

1. Check 3, 4, and 5 as sides of a right triangle and in formula 2.

2. Do the same for 5, 12, and 13.

3. With 6, 8, and 10, check as sides of a right triangle and in formula 1.

4. Try 4, 5, and 6 as sides for a right triangle and in formula 1.

5. May any three numbers be used as sides of a right triangle? Why?

6. In the formula, $c^2 = a^2 + b^2$ (§ 321), if any two of the values are known, can the third value be found? How?

326. Hero's Formula. (Area of a triangle.) If a, b, and c are the sides of any triangle and $s = \frac{1}{2}(a + b + c)$, the area of the triangle equals $= \sqrt{s(s-a)(s-b)(s-c)}$. This is called *Hero's Formula* because it was discovered by Hero of Alexandria. (See p. 386.)

Exercises. Area of a Triangle by Hero's Formula

1. Find the area of a triangle whose sides are 39, 42, and 45.

Solution: $s = \frac{1}{2}(a + b + c) = \frac{1}{2}(39 + 42 + 45) = 63.$

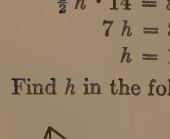
$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{63 \cdot 24 \cdot 21 \cdot 18} =$$
$$\sqrt{9 \cdot 7 \cdot 8 \cdot 3 \cdot 7 \cdot 3 \cdot 9 \cdot 2} = 9 \cdot 7 \cdot 3 \cdot 4 = 756.$$

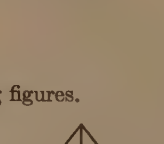
2. Find the area of a triangle whose sides are 10, 17, 21; 26, 35, 51; 26, 28, 30; 75, 176, 229.

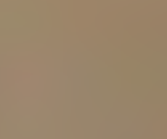
3. The sides of a triangle are 13, 14, and 15. Find the altitude upon the side 14.

Solution: Area $= 84$, by Hero's Formula.
Area $= \frac{1}{2}\,hb$, § 242.
$\frac{1}{2}\,hb = 84$, by substitution.
$\frac{1}{2}\,h \cdot 14 = 84$,
$7\,h = 84$.
$h = 12$.

4. Find h in the following figures.

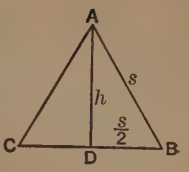






327. Formulas of the Equilateral Triangle

Let h be an altitude of the equilateral triangle ABC, whose sides are s. $CD = DB = \frac{1}{2}s$. Why?



Solution:

In rt. $\triangle ADB$, $h^2 + \left(\frac{s}{2}\right)^2 = s^2$. Area $= \frac{1}{2}hb$.

$h^2 + \frac{s^2}{4} = s^2$. $= \frac{1}{2}s \times \frac{s\sqrt{3}}{2}$.

$h^2 = \frac{3}{4}s^2$. $= \frac{s^2\sqrt{3}}{4}$.

$h = \frac{s}{2}\sqrt{3}$. $= \left(\frac{s}{2}\right)^2\sqrt{3}$.

Exercises. The Equilateral Triangle

1. Find the altitude of an equilateral triangle if a side is 4; 6; 8; 5; 7.

2. Find the area of an equilateral triangle if a side is 4; 6; 8; 5.

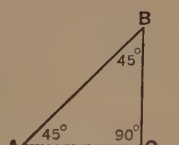
3. Find the side of an equilateral triangle if its altitude is $10\sqrt{3}$; $16\sqrt{3}$; 4; 6.

4. Find the side of an equilateral triangle if its area is $25\sqrt{3}$; $36\sqrt{3}$; 173.2. ($\sqrt{3} = 1.732$.)

5. If h is the altitude of an equilateral triangle and s the side, find s in terms of h. Find s in terms of the area.

6. A regular hexagon whose sides are 8 is divided into six equilateral triangles whose sides are also 8. Find the area of the hexagon.

328. Formulas of the 45–45–90 Triangle



ABC is an isosceles right triangle with $\angle A = \angle B = 45°$. Let s be one of the equal legs.

$(AB)^2 = (BC)^2 + (AC)^2$, by the Pythagorean Theorem.

$= s^2 + s^2$, by substitution.

$= 2s^2$.

$AB = s\sqrt{2}$. (That is, the hypotenuse equals a leg times $\sqrt{2}$).

Area $= \frac{1}{2}s^2$. Why?

Exercises. The 45–45–90 Triangle

1. In an isosceles right triangle, find the hypotenuse if a leg is 4; find the hypotenuse if a leg is $5\sqrt{2}$; find a leg if the hypotenuse is $6\sqrt{2}$; find a leg if the hypotenuse is 6.

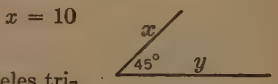
2. In an isosceles right triangle, find the area if a leg is 4; find the area if the hypotenuse is $8\sqrt{2}$; find the area if the hypotenuse is 8; find the hypotenuse if the area is 32.

3. Express the ratio of the hypotenuse to a leg of an isosceles right triangle as a common fraction and then as a decimal to three places; a leg to the hypotenuse; a leg to the other leg.

4. If the bisector of the right angle of an isosceles right triangle is 10, find each leg, the hypotenuse, and the segments of the hypotenuse.

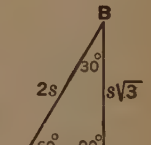
5. What kind of triangles are formed by drawing a diagonal of a square? How does the formula for the diagonal of a square compare with the formula for the hypotenuse of an isosceles right triangle?

6. Find the projection of x upon y if $x = 10$ and the included angle is 45°?



7. If the angle at the vertex of an isosceles triangle is 90°, find the ratio of the altitude to the base.

329. Formulas of the 30–60–90 Triangle



ABC is a right triangle with $\angle A = 60°$ and $\angle B = 30°$.

Let the side opposite 30° equal s.

$AB = 2s$ by § 158. (That is, the hypotenuse equals twice the shorter leg.)

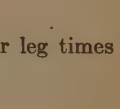
$(BC)^2 = (AB)^2 - (AC)^2$, by § 321.

$(BC)^2 = (2s)^2 - s^2$, by substitution.

$(BC)^2 = 4s^2 - s^2 = 3s^2$.

$BC = s\sqrt{3}$. (That is, the longer leg equals the shorter leg times $\sqrt{3}$.)

Exercises. The 30–60–90 Triangle



1. In the annexed figure, find

a) BC and AC, if $AB = 6$;

b) AB and AC, if $BC = 8$;

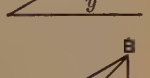
c) BC and AB, if $AC = 4\sqrt{3}$;

d) BC and AB, if $AC = 12$.

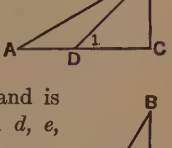
2. Find the area of the triangle in Ex. 1, if $AC = 10$; if $AC = 10\sqrt{3}$; if $BC = 6\sqrt{3}$; if $BC = 6$; if $AB = 8$.

3. In the figure of Ex. 1 express the following ratios, first as common fractions and then as decimals to three places: $\frac{BC}{AB}$; $\frac{AC}{AB}$; $\frac{BC}{AC}$.

4. Find the projection of x upon y if $x = 10$ and the included angle is 60°; 30°.

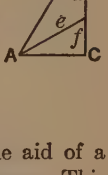


5. In the right triangle ABC, $\angle A = 30°$, $\angle 1 = 45°$, and $BC = 10$. Find all the other parts.



6. In the annexed 30–60–90 triangle, $\angle A = 60°$ and is bisected by line e. The hypotenuse c is 20. Find d, e, and f.

7. How may the formula for the longer leg of the 30–60–90 triangle be made to agree with the formula for the altitude of an equilateral triangle?



TRIGONOMETRIC RATIOS

The properties of right triangles are often studied by the aid of a subject called *trigonometry*. A triangle may be called a *trigon*. This word is derived from two Greek words, *tri* meaning "three" and *gonia* meaning "angle." The Greek word *metron* means "measure." Then trigonometry deals with the *measurement of triangles*.

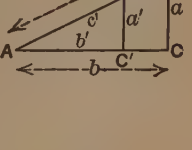
Three Tools so far studied are necessary for the introduction of this subject. They are:

1. The sum of the angles of a triangle equals two right angles. (Tool XI.)

2. The Pythagorean Theorem. (Tool XXV.)

3. Corresponding sides of similar triangles are proportional. (Tool XXIV.)

In the right triangle BCA, from any point such as B' on the hypotenuse, drop the perpendicular $B'C'$. By means of corresponding parts of similar triangles, it may be proved that $\frac{a}{c} = \frac{a'}{c'}$; $\frac{b}{c} = \frac{b'}{c'}$; $\frac{a}{b} = \frac{a'}{b'}$.



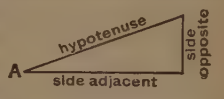
These show that so long as $\angle A$ remains the same, certain ratios of the sides remain the same, that is, these ratios are constant for a given angle. These ratios have special names.

$\frac{a}{c}$ is called the *sine* of the angle A. (*sin A.*)

$\frac{b}{c}$ is called the *cosine* of the angle A. (*cos A.*)

$\frac{a}{b}$ is called the *tangent* of the angle A. (*tan A.*)

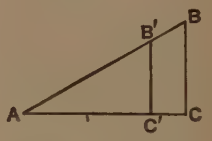
330. Definition of Sine, Cosine, and Tangent. The *sine* of an acute angle is equal to the length of the *side opposite* the angle divided by the length of the *hypotenuse*. The *cosine* of an acute angle is equal to the length of the *side adjacent* to the angle divided by the length of the *hypotenuse*. The *tangent* of an acute angle is equal to the length of the *side opposite* the angle divided by the length of the *side adjacent* to the angle.



Exercises

In the right triangle BCA, $B'C' \perp AC$.

1. Prove $\sin A = \cos B$.
2. Prove $\cos A = \sin B$.
3. Prove $\sin B' = \cos A$.
4. Prove $\sin B = \sin B'$.

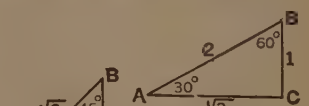


331. Natural Functions. The sine, cosine, and tangent of an angle are sometimes called *natural functions of the angle.* These functions as defined in § 330 are *ratios*, that is, *expressed divisions.* The numbers which correspond to the values of these ratios are abstract numbers and are usually expressed by decimals. They are arranged in tables under their proper headings for different sized angles. Such tables are called *Tables of Natural Functions.* (See page 249.)

Exercise. From the annexed figures show that the following values taken from a table are correct:

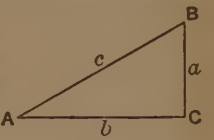
Angle	Sine	Cosine	Tangent
30°	.500	.866	.577
45°	.707	.707	1.000
60°	.866	.500	1.732

HINT: $\sin 30° = \frac{BC}{AB} = \frac{1}{2} = .500$



Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875				
6°	.1045	.9945	.1051	50°	.7660	.6428	1.1918
7°	.1219	.9925	.1228	51°	.7771	.6293	1.2349
8°	.1392	.9903	.1405	52°	.7880	.6157	1.2799
9°	.1564	.9877	.1584	53	.7986	.6018	1.3270
				54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679				
16°	.2756	.9613	.2867	60°	.8660	.5000	1.7321
17°	.2924	.9563	.3057	61°	.8746	.4848	1.8040
18°	.3090	.9511	.3249	62°	.8829	.4695	1.8807
19°	.3256	.9455	.3443	63°	.8910	.4540	1.9626
				64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663				
26°	.4384	.8988	.4877	70°	.9397	.3420	2.7475
27°	.4540	.8910	.5095	71°	.9455	.3256	2.9042
28°	.4695	.8829	.5317	72°	.9511	.3090	3.0777
29°	.4848	.8746	.5543	73°	.9563	.2924	3.2709
				74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002				
36°	.5878	.8090	.7265	80°	.9848	.1736	5.6713
37°	.6018	.7986	.7536	81°	.9877	.1564	6.3138
38°	.6157	.7880	.7813	82°	.9903	.1392	7.1154
39°	.6293	.7771	.8098	83°	.9925	.1219	8.1443
				84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000	90°	1.0000	0.0000	∞

332. Solving a Right Triangle. If in addition to the right angle, a side and any other part are known, the other three parts may be found in a right triangle. The finding of the values of the three unknown parts is called *solving the right triangle.*



If we take the annexed right triangle BCA as a standard triangle and we are given three such parts as described above, in order to find the three remaining parts we shall need, besides a Table of Natural Functions, the following formulas:

$$\angle B + \angle A = 90°;\quad c^2 = a^2 + b^2;\quad \sin A = \frac{a}{c};\quad \cos A = \frac{b}{c};$$

$$\tan A = \frac{a}{b}.$$

Example 1. Given $\angle A = 30°$ and $b = 12$ in., solve the triangle.

Solution:

$A + \angle B = 90°$. $\quad b = 12$ (Given.)

$30° + \angle B = 90°$. $\tan 30° = .577$ from a Table of Natural Functions.

$$\angle B = 60°.\quad \tan 30° = \frac{a}{12}.$$

$$.577 = \frac{a}{12}.$$

$$a = 12 \times .577 = 6.924.$$

$\cos 30° = .866$ from a Table of Natural Functions.

$$\cos 30° = \frac{12}{c}.$$

$$.866 = \frac{12}{c}.$$

$$c = \frac{12}{.866} = 13.856.$$

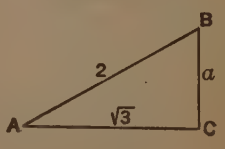
Therefore the remaining parts are $\angle B = 60°$, $a = 6.924$, and $c = 13.856$.

Check: $(13.856)^2 = (12)^2 + (6.924)^2$.

$(191.98) = (144) + (47.94)$.

$191.98 = 191.94$. (*Approximately.*)

Example 2. In the right triangle BCA, if $AC = \sqrt{3}$ and $AB = 2$, solve the triangle.



Solution: $\cos A = \frac{\sqrt{3}}{2} = \frac{1.732}{2}$.

$\cos A = .866.$

In a Table of Natural Functions we find in the cosine column that .866 lies opposite 60°. $\angle A = 60°$. $\angle B = 30°$. Why?

$$\sin 30° = .500.$$

$$\sin 30° = \frac{a}{2}.$$

$$.500 = \frac{a}{2}.$$

$$a = 2 \times .500 = 1.000.$$

Therefore the remaining parts are $\angle A = 60°$, $\angle B = 30°$, and $a = 1$.

Check: $(2)^2 = (\sqrt{3})^2 + (1)^2$.

$4 = 3 + 1$.

$4 = 4$.

Exercises. Trigonometric Ratios

1. With the aid of an isosceles right triangle, compute the value of the sine, cosine, and tangent of 45°. (Use no table.)

2. With the aid of the 30–60–90 triangle, find the value of the sine, cosine, and tangent of 30°. (Use no table.)

3. Find the value of the sine, cosine, and tangent of 60°. (Use no table.)

4. The sides of a right triangle are 5, 12, and 13. Find the two acute angles. (Use Table of Natural Functions, p. 249.)

5. By the use of a table, express the numbers .4067 and .6157 both as sines and cosines.

6. Find the area of an isosceles triangle whose equal sides are 20 in. and include an angle of 80°.

HINT: By trigonometry find the altitude.

7. The altitude upon the hypotenuse of a right triangle is 12 and one leg is 15. Find all the acute angles of the figure.

8. The altitude of a triangle is 8 in. and divides the vertex angle into two angles of 40° and 30°. Find all the remaining parts of the figure.

9. In the annexed figure, if $AD = 10$, $\angle B = 50°$, and $\angle C = 30°$, find AB, AC, and BC.

10. If the tangents from a point to a circle are each 15 in. long and include an angle of 78°, find the radius of the circle.

11. A, B, C, D, and E are five "equally spaced" points about a circle whose radius is 12. Find AB.

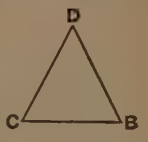
12. Find the angles of a trapezoid whose altitude is 12 and whose legs are 15 and 13, and whose bases are 25 and 39.

13. Find the area of an equilateral triangle whose perimeter is 48.

14. To determine the width of a stream, a surveyor measures a line AB, 350 ft., along one bank. At B he turns a right angle and his assistant places a rod in the line of sight at a point C on the opposite bank of the stream. The angle BAC measures 62°. How wide is the stream?

15. Find the angle of the sun's altitude when the shadow of a man equals twice his height.

333. Drawing to Scale. Suppose $\triangle BCD$ is a correct representation of a triangular flower bed. From the figure alone can you form an idea of its shape? of its size? Can you tell whether the flower bed is large or small? If you knew that the length of BC in the flower bed is 96 times its length in the drawing, how would you find its actual length? BC in the drawing is one-half inch long and we say the scale is $1'' = 96'' = 8'$. This is another way of saying that the sides of the similar figures are proportional and that the ratio of similitude is 1 : 96.



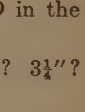
Exercises. Drawing to Scale

1. With the aid of a ruler find the length of CD and BD in the flower bed.

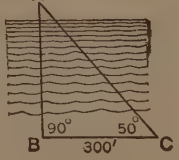
2. If the scale is $1'' = 8'$, what length is represented by $2''$? $3\frac{1}{4}''$? $\frac{3}{4}''$? $\frac{3}{16}''$?

3. If the above scale is used, how long a line will represent $60'$? $168'$? $18'$? $36''$?

4. B is 150 mi. north of A, and C is 60 mi. northeast of A. Draw the figure to a convenient scale and determine the approximate distance in miles between B and C by measurement.



5. Draw $\triangle ABC$ using a convenient scale. Find AB by measurement and calculation. Can this method always be used to find the distance across a river? Explain how the method can be practically applied.

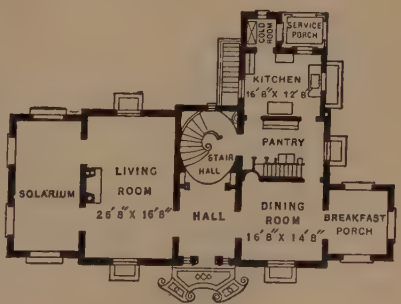


6. The lengths of the sides of a right triangle are 80″, 192″, and 208″. Make a drawing of the triangle using the scale $\frac{1}{4}'' = 16''$.

7. On the map of the State of Texas the scale is 1″ = 320 mi. How many miles does each $\frac{1}{16}''$ represent? Find the approximate distances in an air line from Austin to each of four other cities on the map.

8. Calculate the scale used in drawing the floor plan shown. This may be done from any one of several measurements. Select any two of them and compare your results.

9. After checking your scale in Ex. 8, find the length and the width of the solarium.



10. In a group of 80 pupils, 8 received a grade of A, 20 a grade of B, 32 a grade of C, and the remainder failed. Make a bar graph of these facts, using $\frac{1}{16}''$ as a unit.

Courtesy U. S. Bureau of Education

THE FARM ON THE SAND TABLE

If a surveyor were to make a sketch of a farm, he would take a great many measurements so that all parts would appear in the same scale. All the objects shown above are not in the same scale, and for this reason the cow could not get into the barn.

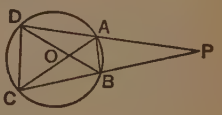
SUPPLEMENTARY EXERCISES

Exercises. Similar Triangles

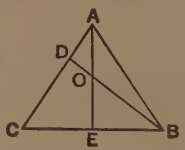
1. A line which bisects two sides of a triangle forms a second triangle similar to the first.

2. Two isosceles triangles are similar if the angle at the vertex of one triangle equals the angle at the vertex of the other.

3. In the annexed figure, prove four pairs of triangles similar.

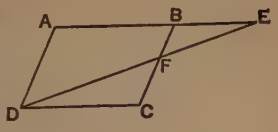


4. In the annexed figure, AE and BD are altitudes intersecting at O. Prove $\triangle AOD \sim \triangle BCD$.



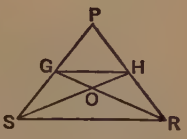
5. The bisectors of two

corresponding angles of similar triangles form two other pairs of similar triangles.



6. In the parallelogram $ABCD$, a line from D intersects BC at F and meets AB produced at E. Prove that three triangles in the figure are similar.

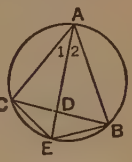
7. In the isosceles triangle PRS two medians, SH and RG, intersect at O.



a) Prove $\triangle GOH \sim \triangle ROS$.
b) Prove $\triangle PGH \sim \triangle PSR$.
c) Prove $\triangle GOS \sim \triangle HOR$.

8. In $\triangle ABC$ if perpendiculars are drawn to AB at A and B meeting AC produced at D and BC produced at E, prove that triangles ACE and BCD are similar.

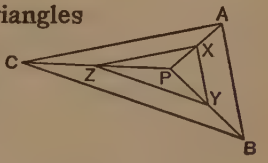
9. If P, R, and S are the mid-points of the sides of ABC, prove $\triangle RSP \sim \triangle ABC$.



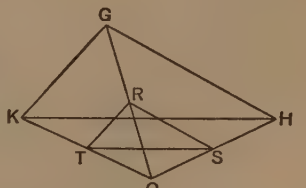
10. In the annexed figure $\angle 1 = \angle 2$.
a) Prove $\triangle ACD \sim \triangle AEB$.
b) Prove $\triangle ABD \sim \triangle AEC$.
c) Prove $\triangle ADC \sim \triangle BDE$.

Exercises. Similar Triangles

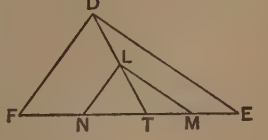
11. In the annexed figure, P is any point within the triangle ABC and X, Y, and Z are the mid-points of PA, PB, and PC. Prove four pairs of triangles similar.



12. In the annexed figure, O is any point outside the triangle GHK and R, S, and T are the mid-points of OG, OH, and OK. Prove as many pairs of triangles similar as you can.



13. In the annexed figure, T is any point on a side of triangle DEF and L, M, and N are the mid-points of TD, TE, and TF. Prove as many pairs of triangles similar as you can.

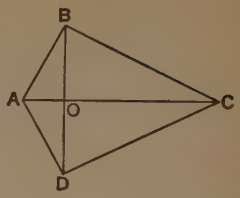


14. In the annexed figure, $AC \perp DB$ and $DO = OB = 2\,OA = \frac{1}{2}\,OC$.

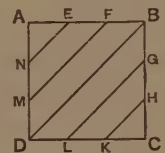
a) Prove $\triangle BOA \sim \triangle BOC$.

b) Prove $\triangle BOA \sim \triangle COD$.

c) Prove $\triangle DOA \sim \triangle BOA$.

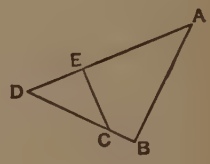


15. In the annexed figure, BD is the diagonal of square $ABCD$ and E, F, G, H, etc. are trisection points of the sides. How many similar triangles are formed? Prove $\triangle AEN \sim \triangle CGL$.



16. In the annexed figure, $AE = 2\,ED = 4\,BC = \frac{4}{3}\,CD$. Prove $\triangle ADB \sim \triangle CDE$.

HINT: Let $BC = X$.



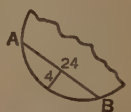
Exercises. Intersecting Chords

17. The five chords in the figure are perpendicular to the diameter and divide it into six equal parts. Find the lengths of the chords if the diameter is 12 in.

18. If P is 5 in. from the center and the segments of a chord through P are 8 and 18, find the diameter of the circle.

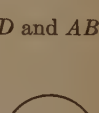
HINT: $(r + 5)(r - 5) = 8 \times 18$.

19. A mechanic has only one fragment of a broken wheel and wishes to find its original diameter. He measures a chord AB and its perpendicular bisector h, finding them to be 24″ and 4″ respectively. What was the diameter of the wheel?



20. If the diameter of the wheel in the previous exercise is D and AB is called c, prove the following formula, $D = \dfrac{c^2 + 4\,h^2}{4\,h}$.

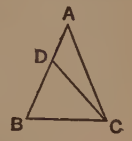
21. Using the formula of Ex. 20, find the diameter of an arch above a window which is 36 in. wide, if the rise of the arch above the horizontal is 4 in.



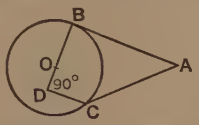
22. A chord 6 in. long is 4 in. distant from the center. Find the diameter of the circle.

Exercises. Proportions and Products from Similar Triangles

23. In isosceles triangle ABC, CD is drawn from an extremity of base BC equal to the base. Prove $(BC)^2 = BD \times BA$.

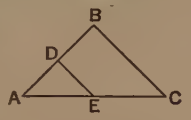


24. In the annexed figure, if AB and AC are tangents, prove $CD : BD = OB : AB$.



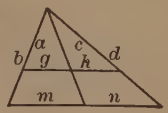
25. If ABC is an inscribed triangle with an altitude AD and a diameter AE, prove $AB : AD = AE : AC$.

26. In the annexed figure, $AB = BC$ and DE is the perpendicular bisector of AB. Prove $AC : AB = AB : AE$.



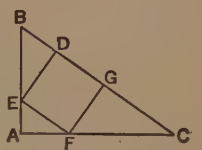
27. In similar triangles the radii of the inscribed circles have the same ratio as two corresponding sides.

28. A median to the base of a triangle bisects any line drawn between the other two sides and parallel to the base.



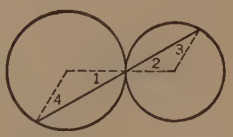
HINT: $\frac{a}{b} = \frac{g}{m}$; $\frac{c}{d} = \frac{h}{n}$; $\frac{a}{b} = \frac{c}{d}$.

29. In right triangle ABC is inscribed a square $DEFG$ with its vertices E and F in the legs AB and AC and the other vertices D and G in the hypotenuse BC. Prove DE a mean proportional between BD and CG.



HINT: $\triangle EDB \sim \triangle CGF$.

30. Two circles are tangent externally and through the point of contact a secant is drawn. The chords formed are proportional to the radii.



HINT: $\angle 1 = \angle 2 = \angle 3 = \angle 4$.

Exercises. The Pythagorean Theorem

31. The legs of a right triangle are $a^2 - b^2$ and $2ab$. Find the hypotenuse.

32. Find the sides of a right triangle if they are three consecutive numbers.

33. A diameter 35 in. long bisects a chord $33\frac{3}{5}$ in. long. Find the distances from the ends of the chord to the ends of the diameter.

34. The length of the common chord of two intersecting circles is 16 and their radii are 10 and 17. Find the distance between their centers.

35. If the radii of two concentric circles are 10 in. and 8 in., find a chord of the larger circle tangent to the smaller.

36. g and h are parallel chords whose distance apart is d. Find the radius of the circle if the center lies on the same side of both chords and $g = 18$, $h = 24$, $d = 3$.

37. p and q are parallel chords whose distance apart is d. Find the radius of the circle if the center lies between the chords and $p = 16$, $q = 12$, $d = 14$.

38. Find the length of a chord if its distance from the center is 6 and the radius of the circle is 10.

39. Find the legs of a right triangle if their ratio is 3 : 4 and the hypotenuse is 40.

40. Find the perimeter of a rhombus whose diagonals are 6 and 8; 18 and 24.

41. How long is the line connecting a lower corner of a room with the opposite upper corner, if the dimensions of the room are 12 by 16 by 9?

42. The non-parallel sides of a trapezoid are each 53 in. long. One of the parallel sides is 56 in. longer than the other. (*a*) Find the altitude of the trapezoid; (*b*) Can you find the area? Why?

SUMMARY OF CHAPTER FOUR

Triangles are similar, if they

(1) are mutually equiangular;

(2) have their respective sides parallel;

(3) have their respective sides perpendicular;

(4) have their respective sides proportional;

(5) have an angle of the one equal to an angle of the other and the including sides proportional;

(6) are similar to the same triangle;

(7) are corresponding triangles of similar polygons.

Polygons are similar, if they

(1) are mutually equiangular and have their corresponding sides proportional;

(2) are composed of like-placed similar triangles.

In similar polygons

(1) corresponding angles are equal;

(2) corresponding sides are proportional;

(3) corresponding sides lie opposite equal angles;

(4) the perimeters have the same ratio as corresponding sides;

(5) the areas have the same ratio as the squares of corresponding sides;

(6) corresponding sides have the same ratio as the square roots of the areas.

In a circle

(1) the products of the segments of two intersecting chords are equal;

(2) a tangent from an external point is the mean proportional between a secant from the same point and its external segment;

(3) the perpendicular from any point on the circle to the diameter is the mean proportional between the segments of the diameter, and

(4) either chord from the point to an end of the diameter is the mean proportional between the diameter and its projection upon the diameter.

In a right triangle

(1) the altitude upon the hypotenuse forms three pairs of similar triangles;

(2) the altitude upon the hypotenuse is the mean proportional between the segments of the hypotenuse; and

(3) either leg is the mean proportional between the hypotenuse and its projection upon the hypotenuse;

(4) the square *of* the hypotenuse equals the sum of the squares *of* the legs;

(5) the square *on* the hypotenuse equals the sum of the squares *on* the legs.

Special Formulas

1. Any triangle: Area $= \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$.

2. Right triangle: $c^2 = a^2 + b^2$.

3. Equilateral triangle with side $= s$: Altitude $= \frac{s\sqrt{3}}{2}$; Area $= \frac{s^2\sqrt{3}}{4}$.

4. Isosceles right triangle with legs $= s$: Hypotenuse $= s\sqrt{2}$; Area $= \frac{1}{2}s^2$.

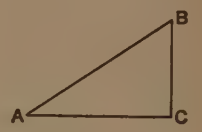
5. 30–60–90 triangle with shorter leg $= s$: Hypotenuse $= 2s$; longer leg $= s\sqrt{3}$.

6. For $\angle A$ of rt. $\triangle BCA$,

$$\text{sine } A = \frac{\textit{side opposite}}{\textit{hypotenuse}} = \frac{BC}{AB};$$

$$\text{cosine } A = \frac{\textit{side adjacent}}{\textit{hypotenuse}} = \frac{AC}{AB};$$

$$\text{tangent } A = \frac{\textit{side opposite}}{\textit{side adjacent}} = \frac{BC}{AC}.$$



REVIEW ORAL EXERCISES. DIAGNOSTIC TEST

1. What two conditions make polygons similar?

2. What is the most common method of proving triangles similar?

3. What special kinds of triangles are similar?

4. What special kinds of quadrilaterals are similar?

5. If two altitudes of a triangle are drawn, how many similar right triangles are formed?

6. If two polygons are mutually equiangular, are they similar?

7. If two triangles are mutually equiangular, are they similar?

8. If two polygons have their corresponding sides proportional, are they similar?

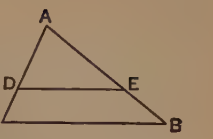
9. If two triangles have their corresponding sides proportional, are they similar?

10. If a and a' denote the altitudes and b and b' the bases of two similar triangles, how may the four quantities be used in a proportion?

11. If a and a' denote the altitudes and b and b' the bases of two equivalent triangles, how may the four quantities be used in a proportion?

12. What is the ratio of the areas of two triangles whose sides are 8, 10, 12 and 20, 25, 30?

13. If $DE \parallel BC$, what is the ratio of the altitudes of $\triangle ADE$ and $\triangle ABC$ when the ratio of their areas is 1 : 4; 1 : 9; 16 : 25; 72 : 98?

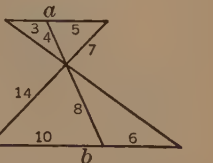


14. In the figure of Ex. 13, if $DE \parallel BC$, what is the ratio of AD to AC if (*a*) DE divides $\triangle ABC$ into two equal parts?

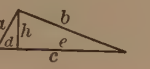
b) $\triangle ADE = \frac{1}{3} \triangle ACB$?

c) $\triangle ADE$: trapezoid $DEBC = 3 : 4$?

15. If $a \parallel b$, what proportion may be written beginning with 3? with 4? with 7? with 10? with 8?



Exercises 16–19 are based upon the right triangle *abc* with *h* perpendicular to *c*.



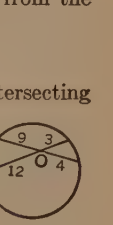
16. How many pairs of similar triangles may be found in the figure? Name them.

17. What three lines are mean proportionals? Write the proportions.

18. Write three applications of the Pythagorean Theorem from the figure.

19. State two methods of finding the area of *abc*.

Exercises 20–23 are based upon the circle with chords intersecting at *O*.



20. What theorem does the figure suggest?

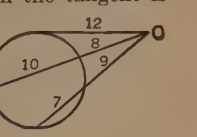
21. What equality may be written from the figure?

22. What proportions may begin with 3?

23. What is the product of the segments of any other chord through *O*?

Exercises 24–27 are based upon the circle with a tangent and two secants drawn from *O*.

24. Write two proportions from the figure in which the tangent is the mean proportional.



25. Write an equality involving both secants.

26. If another secant is drawn through *O* with 10 as its external segment, what is the length of the secant?

27. If the radius of the circle is 5, what is the length of the longest secant through O?

28. If one equilateral triangle has a perimeter four times that of another, how do their areas compare?

29. If one equilateral triangle has an area four times that of another, how do their perimeters compare?

30. If the altitude of an isosceles triangle is doubled, how is the area affected?

31. If the altitude of an equilateral triangle is doubled, how is the area affected?

32. The altitude of a triangle is 5. What is the altitude of a similar triangle (*a*) whose perimeter is 9 times as large? (*b*) whose area is 9 times as large?

33. The side of an equilateral triangle is 6. What is the side of an equilateral triangle $2\frac{1}{4}$ times as large?

34. The sides of a parallelogram are 5 and 8 and the projection of 5 upon 8 is 3. What is the area?

35. If a rhombus has a perimeter of 24, is its area increased if one of the angles is increased from 60° to 120°?

MISCELLANEOUS EXERCISES

1. The difference of the squares on two sides of a triangle equals the difference of the squares on the segments of the third side formed by the altitude to the third side.

2. The gable end of a house is right-angled and its width is 30 ft. Find (*a*) the height of the gable and (*b*) the length of each of the rafters.

3. The legs of a right triangle are 21 and 28. (*a*) Find the lengths of the segments into which the bisector of the right angle divides the hypotenuse; (*b*) Find the area of the three triangles in the figure.

4. A ladder 32 ft. 6 in. long when stood on a street reaches a window 26 ft. above the street on one side and a window 16 ft. 6 in. above the street on the other side. Find the width of the street.

5. If three similar triangles are drawn with the sides of a right triangle as their corresponding sides, the triangle upon the hypotenuse equals the sum of the triangles upon the legs.

6. Find the legs of a right triangle if their projections upon the hypotenuse are 9 and 16; if their projections are 2.88 and 5.12.

7. Two triangles are similar and the sides of the smaller are 6, 8, and 10. Find the remaining sides of the larger if its shortest side is 9.

8. Two triangles are similar and the sides of the smaller are 8, 14, and 18. Find the sides of the larger if its perimeter is 60.

9. Similar polygons are to each other as the squares of two corresponding diagonals.

10. Mutually equiangular parallelograms have the same ratio as the products of two adjacent sides.

11. The areas of two similar triangles are to each other as the squares of two corresponding medians.

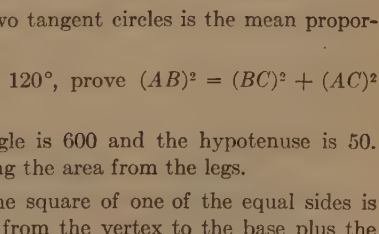
12. The areas of two similar triangles have the same ratio as the squares of the bisectors of two corresponding angles.

13. The equal sides of an isosceles triangle are each 10 in. Find the length of the base if the equal angles are each 45°; 60°; 30°.

14. In drawing a map of a field with sides 75 rd., 60 rd., 50 rd., and 40 rd., the longest side is made 3 in. long. Find the lengths of the other three sides of the drawing. Check your answer by finding the ratio of the perimeters of the field and of the drawing.

15. In the annexed figure, $\angle x = \angle x = \angle x = 45°$, and $BC = BA$. Prove $CA : EA = EA : DE$.

HINT: BE and BC are bisectors respectively of an interior angle and an exterior angle of $\triangle ABD$.



16. The external tangent of two tangent circles is the mean proportional between the diameters.

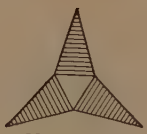
17. If $\angle C$ of $\triangle ABC$ equals 120°, prove $(AB)^2 = (BC)^2 + (AC)^2 + AC \times BC$.

18. The area of a right triangle is 600 and the hypotenuse is 50. Find the legs and check by finding the area from the legs.

19. In an isosceles triangle, the square of one of the equal sides is equal to the square on any line from the vertex to the base plus the product of the segments of the base.

20. In a right triangle, a perpendicular is drawn from the vertex of the right angle to the hypotenuse. Show that the areas of the two triangles thus formed have the same ratio as the squares of the legs of the original triangle.

5TH CORPS



3RD CORPS

88TH DIVISION

U. S. ARMY INSIGNIA

39TH DIVISION

83RD DIVISION

27TH DIVISION

Geometric figures frequently form the foundation for many well known trade-marks, emblems, insignia, etc.

CHAPTER FIVE

REGULAR POLYGONS—THEIR CIRCLES

REGULAR POLYGONS AND THEIR CIRCLES

334. Regular Polygons. In section 160 a regular polygon was defined as one that is both *equiangular* and *equilateral.* An equilateral triangle and a square are regular polygons of three and four sides respectively. Other regular polygons usually studied in elementary geometry are those which have five, six, eight, or ten sides. It was proved by Gauss (p. 386) that only a limited number of regular polygons may be found by elementary geometry; we do not discuss, for example, regular polygons of seven, nine, or eleven sides.

Proposition 1 (§ 335) which follows *assumes that a circle has been divided* into three or more equal arcs. The actual division cannot be made by elementary methods except in certain cases, some of which will be discussed later.

Oral Exercises

1. What is an inscribed angle? How may the size of such an angle be measured? What is meant by the expression that an angle equals numerically its intercepted arc?

2. If an inscribed angle and an angle formed by a tangent and a chord intercept the same or equal arcs of a circle, are the angles equal? Why?

3. The sides of a circumscribed polygon are what special lines of the circle? The sides of the inscribed polygon are what special lines of the circle?

4. The sum of the interior angles of a polygon is equal to what?

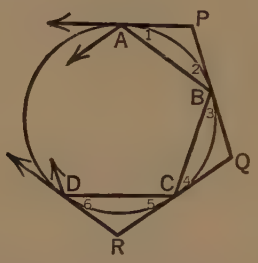
5. Each interior angle of a regular polygon of n sides equals what?

Proposition 1 (c N)

335. Theorem. *If a circle is divided into three or more equal arcs,*

(1) *the chords of these arcs form a regular inscribed polygon;*

(2) *the tangents at the points of division form a regular circumscribed polygon.*



Given (1): A circle with $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} =$ etc. and chords AB, BC, CD, etc.

To prove (1): $ABCD \ldots . A$ is a regular inscribed polygon.

HINT: Prove the polygon equiangular and equilateral.

Proof (1): STATEMENTS	FACTS
1. $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD}$ = etc.	1. Given.
2. If there are n equal arcs, $\angle A$, $\angle B$, $\angle C$, etc., are each equal numerically to half the sum of $(n - 2)$ equal arcs.	2. § 224.
3. $\angle A = \angle B = \angle C$ = etc.	3. Sub.
4. $AB = BC = CD$ = etc.	4. § 196.
5. $ABCD \ldots A$ is regular.	5. Def.

Given (2): A circle with $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD}$ = etc. and tangents at A, B, C, etc.

To prove (2): $PQR \ldots . P$ is a regular circumscribed polygon.

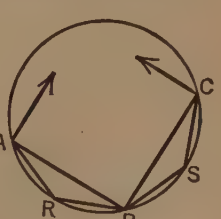
HINT: What two conditions make a polygon regular?

Proof (2):

STATEMENTS	FACTS
1. $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD}$ = etc.	1. Given.
2. $AB = BC = CD$ = etc.	2. Why?
3. $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, etc., are each equal numerically to half $\overset{\frown}{AB}$ or its equal.	3. Why?
4. $\angle 1 = \angle 2 = \angle 3 = \angle 4$ = etc.	4. Why?
5. $\triangle PAB \cong \triangle QBC \cong$ etc.	5. *a.s.a.* = *a.s.a.*
6. $AP = BQ$ and $PB = QC$, etc.	6. § 77
7. $AP = PB$ and $BQ = QC$, etc.	7. § 120.
8. $AP = PB = BQ = QC$ = etc.	8. Sub.
9. $PQ = QR$ = etc.	9. *Prin.* 5.
10. $\angle P = \angle Q = \angle R$ = etc.	10. § 77.
11. $PQR \ldots . P$ is regular.	11. Def.

336. Corollary 1. *An equilateral polygon inscribed in a circle is regular.*

337. Corollary 2. *Chords drawn from each vertex of a regular inscribed polygon to the mid-points of the adjacent arcs form a regular inscribed polygon of double the number of sides; tangents drawn at the mid-points of the arcs cut off by the sides of a regular circumscribed polygon form a regular circumscribed polygon of double the number of sides.*

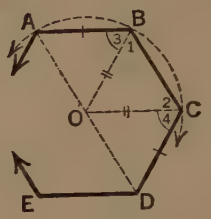


Exercises

1. How many degrees are in each angle of a regular pentagon?
2. Prove that the diagonals of a regular pentagon are equal.

Proposition 2 (C)

338. Theorem. *A circle may be circumscribed about a regular polygon.*



Given: $ABCDE$ . . . A is a regular polygon.

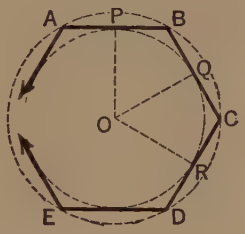
To prove: A circle may be circumscribed about the polygon.

HINT: Show that the circle through any three vertices passes through all the other vertices. Prove $OD = OA$ (a radius).

Proof: STATEMENTS	FACTS
1. Let a circle be drawn through any three vertices such as A, B, and C.	1. § 217.
2. Draw OA, OB, and OC, radii of the circle. Draw OD.	2. *Prin.* 13.
In $\triangle OAB$ and $\triangle ODC$,	
3. $AB = CD$.	3. Why?
4. $OB = OC$.	4. Why?
5. $\angle 1 = \angle 2$.	5. Why?
6. $\angle B = \angle C$.	6. Why?
7. $\angle 3 = \angle 4$.	7. Why?
8. $\triangle OAB \cong \triangle ODC$.	8. *s.a.s.* = *s.a.s.*
9. $OA = OD$.	9. Why?
10. D is upon the circle and in like manner E and all the other vertices are upon the circle.	10. Def.

Proposition 3 (c)

339. Theorem. *A circle may be inscribed in a regular polygon.*



Given: *ABCDE . . . A* is a regular polygon.

To prove: A circle may be inscribed in the polygon.

HINT: Draw a circumscribed circle. What is true of the equal chords of a circle?

Proof:

STATEMENTS	FACTS
1. Let a circle be circumscribed about the polygon.	1. § 338.
2. *AB*, *BC*, *CD*, etc., are equal chords.	2. Given.
3. Draw *OP*, *OQ*, *OR*, etc., the perpendiculars from the center.	3. *Prin.* 15.
4. $OP = OQ = OR =$ etc.	4. Why?
5. *OP*, *OQ*, *OR*, etc., are the radii of a circle inscribed in the polygon.	5. Def.

340. Corollary. *The circles circumscribed about and inscribed in a regular polygon have a common center.*

341. Center of a Regular Polygon. The common center of the circles circumscribed about and inscribed in a regular polygon is called the *center of the polygon*, as O in the annexed figure.

342. Radius of a Regular Polygon. The radius of the circumscribed circle is called the *radius of the polygon*, as OE or OA.

343. Apothem of a Regular Polygon. The radius of the inscribed circle is called the *apothem of the polygon*, as OH.

344. Angle at the Center. The angle formed by two consecutive radii of a regular polygon is called the *angle at the center* of the polygon, as $\angle AOF$ or $\angle FOE$.

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TREASURES OF THE SNOW

345. Corollary 1. *The central angle of a regular polygon of n sides equals* $\frac{360°}{n}$.

346. Corollary 2. *A chord whose length is that of a radius of a circle has an arc of 60° and is a side of a regular inscribed hexagon.*

347. Corollary 3. *Regular hexagons can be inscribed in a circle and circumscribed about a circle.*

HINT: Let OA be a radius of a given circle O. With A as a center and radius AO draw another circle intersecting the first at B. Then arc $AB = 60°$. Apply § 335.

348. Corollary 4. *The radius of a regular polygon bisects an angle of the polygon, and the apothem of a regular polygon is the perpendicular bisector of a side.*

Oral Exercises

1. May a polygon be equiangular, but not equilateral? equilateral, but not equiangular? equiangular and equilateral? Illustrate.

2. May a circle be circumscribed about any polygon?

3. If the diagonal of a square is 5 in., what is the radius of the circumscribed circle?

4. The side of a square is s, what is its apothem?

5. How much larger is the central angle of a regular polygon of 10 sides than the central angle of an equilateral triangle.

6. Find the central angle of a regular pentagon; a regular hexagon; a regular octagon; a regular decagon.

7. How many sides has a regular polygon whose central angle is 10°?

8. The diameter of a circle is 10 in. Find the perimeter of a regular inscribed hexagon.

9. If s is the side of a regular hexagon, what is its apothem?

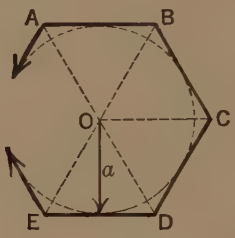
10. If a is the apothem of a square, what is its radius?

11. If a is the apothem of an equilateral triangle, what is its radius?

12. If a polygon is regular, may a circle be inscribed in it? If so, how is the center found? What is the radius of the circle? What is the relation of the radius to each side of the polygon?

Proposition 4 (C N)

349. Theorem. *The area of a regular polygon equals half the product of its apothem by its perimeter.*



Given: $ABCD \ldots A$ is a regular polygon of n sides with A the area, p the perimeter, and a the apothem.

To prove: $A = \frac{1}{2}ap$.

HINT: Divide the polygon into triangles and add their areas.

Proof: STATEMENTS	FACTS
1. A circle may be inscribed with center O and radius equal to a.	1. § 339 and § 343.
2. Draw OA, OB, OC, etc.	2. *Prin.* 13.
3. a is the altitude of each triangle.	3. § 348.
4. $\triangle ODE = \frac{1}{2}a \times ED$. $\triangle ODC = \frac{1}{2}a \times DC$. $\triangle OCB = \frac{1}{2}a \times CB$, etc.	4. Why?
5. $\triangle ODE + \triangle ODC + \triangle OCB \ldots = \frac{1}{2}a(ED + DC + CB + \ldots)$.	5. *Prin.* 4.
6. $\triangle ODE + \triangle ODC + \triangle OCB + \ldots = A$.	6. Why?
7. $ED + DC + CB + \ldots = p$.	7. Why?
8. $A = \frac{1}{2}ap$.	8. Sub.

Exercises. Regular Polygons

1. An angle of a regular polygon is the supplement of the angle at the center.

2. An equiangular polygon circumscribed about a circle is a regular polygon.

HINT: $\triangle MAN \cong \triangle NBO$, etc.

3. How many sides has a regular polygon whose angle at the center is 72°? 30°? 45°?

4. Prove that the side of a square is twice its apothem.

5. Solve each of the following equations for x and check:

a) $a^2 + x^2 = (a\sqrt{2})^2$.

b) $b^2 - \left(\frac{b}{2}\right)^2 = x^2$.

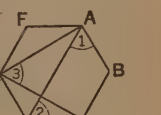
c) $x^2 = r^2 - \left(\frac{r}{2}\sqrt{2}\right)^2$.

d) $x^2 + \left(\frac{a}{2}\sqrt{3}\right)^2 = (2a)^2$.

6. The apothem of an equilateral triangle equals what fractional part of a side?

7. Find the radius of an equilateral triangle whose side is 5 in.

8. If *ABCDEF* is a regular hexagon, prove $\angle 1 = \angle 2 = \angle 3$.



9. In a given circle, the side of an inscribed square is 4 in. Find the diameter of the circle.

10. The diameter of a circle is 6 in. What is the side of the regular inscribed hexagon? Of the regular circumscribed hexagon?

(Supplementary Exercises 1–11, p. 291, may be studied now.)

Oral Exercises

1. If the number of sides of an inscribed regular polygon is doubled, how do the angles of the polygons compare? How do the angles at the center compare?

2. If two perpendicular diameters are drawn in a circle and their ends successively joined, what figure is formed? Prove your answer.

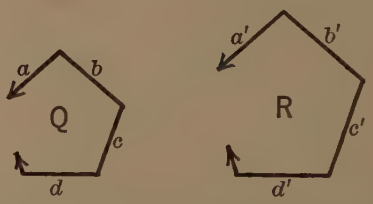
3. For a regular polygon to have eight sides, what must its central angle equal? If it has five sides? seven sides? nine sides?

4. Are the angles of two regular polygons of an unequal number of sides equal? Why?

5. What are the two conditions which must be satisfied in determining regular polygons? In determining similar polygons?

Proposition 5 (c)

350. Theorem. *Regular polygons of the same number of sides are similar.*



Given: Q and R are regular polygons of n sides.

To prove: $Q \sim R$.

HINT: What two conditions make polygons similar?

Proof: STATEMENTS	FACTS
1. Each angle of each polygon equals $\frac{(n-2)}{n}$ st. $\angle$.	1. Why?
2. $a = b = c = d = \text{etc.}$ $a' = b' = c' = d' = \text{etc.}$	2. Why?
3. $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'} = \text{etc.}$	3. Axiom of Division.
4. $Q \sim R$.	4. Def.

Oral Exercises

1. Why are equilateral triangles similar?
2. Are similar hexagons regular?
3. Are regular hexagons similar?
4. May a regular hexagon and a regular pentagon be similar?
5. Are squares with equal sides similar? regular? equal? congruent?
6. The perimeters of regular polygons of the same number of sides have the same ratio as any two corresponding sides. Why?

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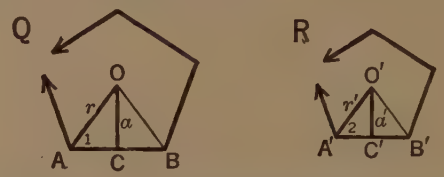
Proposition 6 (c n)

351. Theorem. *The perimeters of regular polygons of the same number of sides have the same ratio as their radii or as their apothems.*



Given: Q and R are regular polygons of n sides with radii r and r', apothems a and a', perimeters p and p', and corresponding sides AB and $A'B'$.

To prove: $\frac{p}{p'} = \frac{a}{a'} = \frac{r}{r'}$.

HINT: Use the Substitution Method. Find some value for $\frac{p}{p'}$.

Proof:

STATEMENTS	FACTS
1. $Q \sim R$.	1. § 350.
2. $\frac{p}{p'} = \frac{AB}{A'B'}$.	2. Why?
3. $AB = 2\,AC$ and $A'B' = 2\,A'C'$.	3. § 348.
4. $\frac{p}{p'} = \frac{AB}{A'B'} = \frac{2\,AC}{2\,A'C'} = \frac{A\,C}{A'C'}$.	4. Why?
5. $\angle A = \angle A'$.	5. §§ 350, 292.
6. $\angle 1 = \angle 2$.	6. § 348.
7. $OC \perp AB$; $O'C' \perp A'B'$.	7. § 348.
8. rt. $\triangle OCA \sim$ rt. $\triangle O'C'A'$.	8. Why?
9. $\frac{AC}{A'C'} = \frac{r}{r'} = \frac{a}{a'}$.	9. Why?
10. $\frac{p}{p'} = \left(\frac{AC}{A'C'}\right) = \frac{r}{r'} = \frac{a}{a'}$.	10. Sub.

352. Corollary. *Regular polygons of the same number of sides have the same ratio as the squares of their radii or as the squares of their apothems, or as the squares of their perimeters.*

Proof: In the previous figure,

STATEMENTS	FACTS
1. $\frac{AC}{A'C'} = \frac{r}{r'} = \frac{a}{a'} = \frac{p}{p'}$.	1 Why?
2. $\frac{AC^2}{A'C'^2} = \frac{r^2}{r'^2} = \frac{a^2}{a'^2} = \frac{p^2}{p'^2}$.	2. Why?
3. $\frac{Q}{R} = \frac{AC^2}{A'C'^2}$.	3. Why?
4. $\frac{Q}{R} = \frac{r^2}{r'^2} = \frac{a^2}{a'^2} = \frac{p^2}{p'^2}$.	4. Why?

Exercises. Ratios of Regular Polygons

1. The sides of two regular pentagons are 4 in. and 6 in. respectively. Find the ratio of their radii; of their apothems; of their areas.

2. Find the ratio of the perimeters of two regular octagons whose areas are 16 sq. ft. and 9 sq. yds. respectively.

3. Find the ratio of the areas of two regular hexagons whose apothems are $a\sqrt{3}$ and $\frac{b}{2}\sqrt{3}$. Check by finding each area.

4. For two regular polygons of the same number of sides, prove that the ratio of their respective sides equals the ratio of their respective apothems.

5. If the radius of a circle is r, find the ratio of the sides of the inscribed and the circumscribed (*a*) squares; (*b*) equilateral triangles; (*c*) regular hexagons.

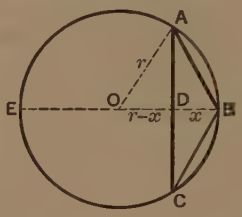
6. Find the ratio of the areas in each part of Ex. 5.

7. Find the ratio of the areas of the square and the equilateral triangle circumscribed about a circle whose radius is R.

8. The perimeters of three regular polygons of the same number of sides have the ratio 3 : 4 : 5. If the area of the middle-sized polygon is 18 sq. in. find the area of each of the other polygons.

9. Given s as the side of a square. Find the area of the square formed by joining the mid-points of the apothems.

353. Problem. If b is a side and r the radius of a regular inscribed polygon, find s, a side of the regular inscribed polygon of double the number of sides.



Given: AC, or b, a side of a regular inscribed polygon of n sides; $AB = s$, a side of a regular inscribed polygon of $2n$ sides; r is the radius.

Required: A value for s in terms of r and b.

HINT: Find a value for s and substitute.

Proof: STATEMENTS	FACTS
1. Draw diameter $EOB \perp AC$.	1. *Prin.* 15.
2. Draw radius OA.	2. *Prin.* 13.
3. $AD = \frac{b}{2}$.	3. § 200.
4. $s^2 = 2rx$.	4. § 319.
In $\triangle ADO$	
5. $(r-x)^2 = r^2 - \left(\frac{b}{2}\right)$.	5. § 321.
6. $(r-x) = \sqrt{\frac{4r^2-b^2}{4}}$.	6. *Prin.* 9.
7. $x = r - \sqrt{\frac{4r^2-b^2}{4}}$.	7. Axiom of Subtraction.
8. $s^2 = 2r\left(r - \frac{\sqrt{4r^2-b^2}}{2}\right)$.	8. Sub. in No. 4.
9. $s = \sqrt{2r^2 - r\sqrt{4r^2-b^2}}$. $= \sqrt{r[2r - \sqrt{(2r-b)(2r+b)}]}$.	9. *Prin.* 9.

Example: If the side of a regular hexagon is 6 in., how long is the side of a regular twelve-sided polygon inscribed in the same circle?

Solution: $S = \sqrt{r\,[2r - \sqrt{(2r-b)(2r+b)}]}$.

$b = 6$, given; $r = 6$ by § 346; $2r = 12$;

$(2r - b) = 6$; $(2r + b) = 18$.

$S = \sqrt{6\,[12 - \sqrt{6 \cdot 18}]} = 6\sqrt{2 - \sqrt{3}}$.

Exercise. Find the side of the regular octagon inscribed in a circle whose radius is 4 in., by using the formula of § 353.

VARIABLES—CONSTANTS—LIMITS

We know of many things which are in a state of change. Plants and animals, for example, increase in size because of growth. A stream of water is continually wearing away its bed. The working parts of a machine are reduced in size because of friction. The panes of glass in our windows vary slightly in size because of changes in temperature.

354. Variable. A quantity which changes in value during a given discussion is called a *variable*. For example, if x takes on the values of 1, 2, 3, 4, etc., in succession, it is a variable and one which *increases without bound;* if x takes on the values of -1, -2, -3, -4, etc., in succession, it is a variable and one which *decreases without bound.*

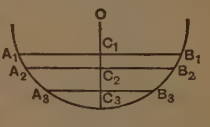
355. Constant. A quantity which does not change in value during a discussion is called a *constant*. For example, three, five, zero, etc., are constants.

There are many cases where a variable does not increase or decrease without bound. Let us consider the positive fraction $\frac{1}{x}$, and have x take the successive values of 1, 2, 3, 4, etc.; the variable $\frac{1}{x}$ then becomes $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, etc., and continues to decrease in value, but not without bound, for it cannot go below the constant zero.

Let us also consider the perimeter of a regular polygon of n sides inscribed in a circle and have n take the successive values of 4, 8, 16, 32, etc. The perimeter continues to increase in size, but not without bound, for it cannot become larger than the circumference of the circle which remains constantly the same size.

356. Limit. When a variable so approaches a constant that the difference between these two quantities becomes and remains less than any previously assigned quantity, however small, the constant is called the *limit* of the variable; and, it will be assumed in the following discussions that *the limit may be substituted in an equation for the variable.*

For example, if a regular polygon inscribed in a circle has the number of its sides increased without bound and the successive sides are represented by A_1B_1, A_2B_2, A_3B_3, etc., then OC_1, OC_2, OC_3, etc., may represent the successive apothems. Here the apothem approaches the radius of the circle as its limit, because the difference between the constant radius and the variable apothem may be made smaller than any previously assigned value, and we assume that the radius may be substituted in an equation for the variable apothem.



The symbol ($\rightarrow$) is often used to designate that a variable approaches a limit. As, apothem $\rightarrow$ radius, which is read, "the apothem approaches the radius as its limit."

357. Area of a Circle. The area of the surface inclosed within a circle is called the area of the circle.

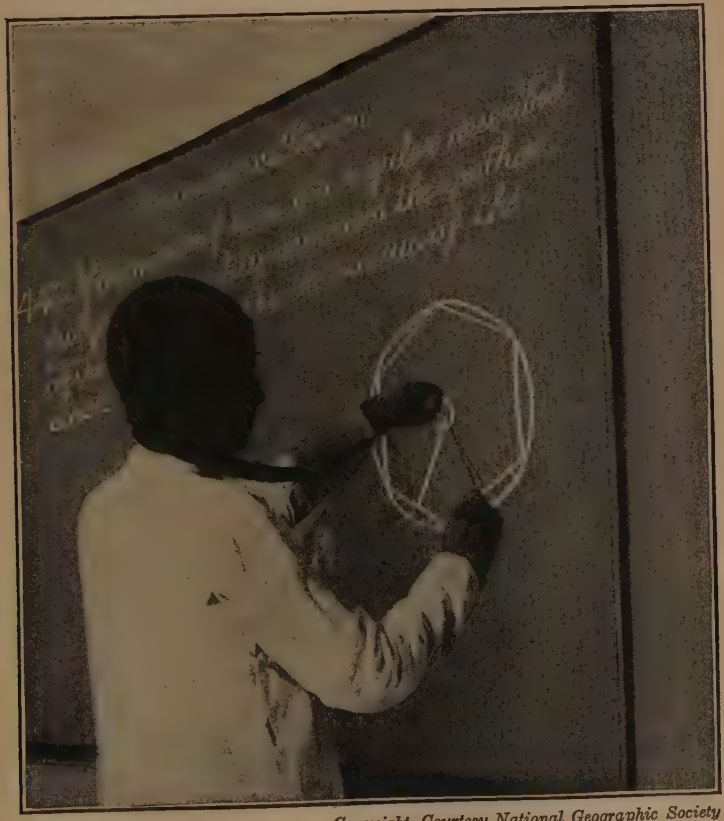
358. Principles of Variables. The following principles are assumed in determining the properties of circles from those of regular polygons. If the *number* of sides of an inscribed or circumscribed regular polygon *be increased without bound:*

Prin. 40. the radius of the circle may be substituted in an equation for the apothem of the inscribed polygon;

Prin. 41. the circumference of the circle may be substituted in an equation for the perimeter of either polygon;

Prin. 42. the area of the circle may be substituted in an equation for the area of either polygon.

NOTE: The apothem of the circumscribed polygon is a constant and is equal to the radius of the circle.



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A LESSON IN GEOMETRY

A boy of the old régime in Peking University used his queue in drawing a circle for his problem. One way to circumscribe a circle about a polygon!

Proposition 7 (c)

359. Theorem. *The circumferences of two circles have the same ratio as their radii, or as their diameters.*

Given: C and c are the circumferences; R and r are the radii; D and d are the diameters of two circles.

To prove: $\frac{C}{c} = \frac{R}{r} = \frac{D}{d}$.

HINT: In the circles inscribe regular polygons of the same number of sides, and let the number of sides be increased without bound. What Principle of Variables refers to circumferences?

Proof:

STATEMENTS	FACTS
1. We may inscribe regular hexagons in the given circles; call their perimeters P and p.	1. § 347.
2. $P : p = R : r$.	2. § 351.
3. If the number of sides of the polygons be increased uniformly so that $P \to C$ and $p \to c$, then $C : c = R : r$.	3. *Prin.* 41.
4. $\frac{R}{r} = \frac{2R}{2r} = \frac{D}{d}$.	4. Why?
5. $\frac{C}{c} = \frac{R}{r} = \frac{D}{d}$.	5. Sub.

360. Corollary. *The ratio of any circumference to its diameter is a constant.*

For $\frac{C}{c} = \frac{D}{d}$, by § 359; $\frac{C}{D} = \frac{c}{d}$ by § 272.

361. The Number Called π. The resulting formula of § 360 means that when any circumference is divided by its diameter, the number obtained is the same as when any other circumference is divided by its diameter; i.e., the number is a constant. This constant which has been named π (pronounced pī) is the number 3.1416 of arithmetic. π is a letter of the Greek alphabet and corresponds to the English letter p. It is the first letter of the Greek word $\pi\epsilon\rho\iota\phi\acute{\epsilon}\rho\epsilon\iota\alpha$, which means "circumference" and when used for the first time, this symbol stood for the boundary of a circle in the same sense as we use C. The present use of π as 3.1416 is due to William Jones (1607) who used it as shown in the following corollary (§ 362).

The investigation of the value of π has claimed the attention of mathematicians for over two thousand years. Probably the earliest approximation was 3, a value used by the Babylonians and ancient Israel. The Talmud says, "What is three hand-breadths around is one hand-breadth through." The Egyptian priest, Ahmes, gave the value of 3.1604. Archimedes, the greatest mathematician of ancient Greece, proved that the value lies between $3\frac{1}{7}$ and $3\frac{10}{71}$. Metius, of Holland, discovered the value of $\frac{355}{113}$ which is correct to six decimal places and which is easily remembered from the sequences of figures 113–355. Shanks, of England, computed the value to 707 decimal places. In advanced mathematics, it is proved that the exact value of π cannot be calculated.

362. Corollary (c N). *If C is the circumference and D the diameter of a circle, $C = \pi D = 2\pi r$.*

For $\frac{C}{D} = \pi$; therefore $C = \pi D = 2\pi r$. Why?

Oral Exercises

1. The radii of two circles are 6 in. and 8 in. respectively. What is the ratio of their diameters? of their circumferences?

2. How much shorter is the diameter than the semi-circumference of a circle whose radius is $3\frac{1}{2}$ in.?

HINT: Use $\pi = \frac{22}{7}$.

Proposition 8 (c N)

363. Theorem. *The area of a circle equals π times the square of the radius.*

Given: O is a circle with radius r; C, the circumference; and A, the area.

To prove: $A = \pi r^2$.

HINT: What Principle of Variables refers to the area of a circle? To the circumference of a circle?

Proof: STATEMENTS	FACTS
1. We may circumscribe a regular hexagon about circle O; call its perimeter p and its area A'.	1. § 347.
2. The apothem $= r$.	2. Def., and see Note § 358.
3. $A' = \frac{1}{2} rp$.	3. § 349.
If the number of sides of the polygon be increased without bound so that $A' \rightarrow A$ and $p \rightarrow C$, then	
4. $A = \frac{1}{2} rC$.	4. *Prins.* 41 and 42.
5. $C = 2 \pi r$.	5. § 362.
6. $A = \frac{1}{2} rC = \frac{1}{2} r \cdot 2 \pi r = \pi r^2$.	6. Sub.

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A SZECHWAN IRRIGATION WHEEL

These Ferris-wheel contrivances are frequently seen along the shores of shallow streams, when the banks are lined with farms. With the exception of the upright supports and the axle, this wheel, some 70 feet high, is constructed entirely of bamboo. Woven bamboo paddles on the rim catch the force of the rapid water in the mill race. Between the paddles are joints of bamboo, open at one end, which scoop up water and empty it into a trough of bamboo poles at the top, which conveys it to the rice fields, 50 to 60 feet above river level.

What parts of equal length can you observe in this picture?

NOTE: Sometimes Proposition 8 (§ 363) is stated in two parts; as, "The area of a circle equals half the product of its radius and circumference," and as a corollary, "$A = \pi r^2$." These two statements are listed respectively as Standard Book Proposition (*C*) and Auxiliary Book Proposition (*c*) by the College Entrance Examination Board.

364. Corollary (c). *The areas of two circles have the same ratio as the squares of their radii, or as the squares of their diameters.*

For $\frac{A}{A'} = \left(\frac{\pi R^2}{\pi r^2}\right) = \frac{R^2}{r^2} = \frac{4R^2}{4r^2} = \frac{(2R)^2}{(2r)^2} = \frac{D^2}{d^2}$. Why?

365. Tool XXVI. **Circumference of a circle $= \pi D = 2\pi R$.**
Area of a circle $= \pi R^2 = \frac{1}{4}\pi D^2$.

Sight Exercises. Circumferences and Areas of Circles

Work the following exercises in terms of π. (Use no numerical value of π.)

1. Find the circumference of a circle whose radius is 3; 4; 5; 6.

2. Find the area of a circle whose radius is 3; 4; 5; 6.

3. Find the radius of a circle if its circumference is 16π; 24π; 32π.

4. Find the radius of a circle if its area is 16π; 49π; 100π.

5. Find the area of a circle whose circumference is 18π; 22π; 40π.

6. Find the circumference of a circle whose area is 16π; 25π; 64π.

7. Find the ratios of the circumferences of two circles whose radii are 2 and 4; 2 and 6; 3 and 6.

8. Find the ratio of the areas of two circles whose radii are 2 and 4; 2 and 6; 3 and 6.

9. A 4″ circle is how much larger than a 2″ circle? (A 4″ circle is a circle whose diameter is 4″.)

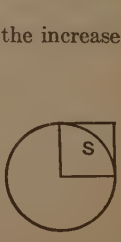
10. What is the area of the ring formed by concentric circles whose diameters are 4 and 6?

11. If 1″ is added to the diameter of a 3″ circle, what is the increase in the circumference? in the area?

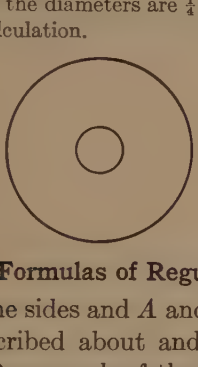
12. How many 2″ circles are equal in area to a 6″ circle?

13. What is the ratio of the area of the accompanying circle to that of the square *S*?

14. What is the ratio of the circumference of the accompanying circle to the perimeter of square *S*?



15. How many of the small circles do you think are required to equal in area the larger circle if the diameters are $\frac{1}{4}$ in. and 1 in. respectively? Test your estimate by calculation.



Exercises. Formulas of Regular Polygons

s and s' represent the sides and A and A' the areas of regular polygons circumscribed about and inscribed in a circle whose radius is R. Prove each of the following statements, if the subscripts designate the number of sides of the polygons:

1. $s'_4 = R\sqrt{2}$.

2. $s_4 = 2R$.

3. $s'_6 = R$

4. $s_6 = \dfrac{2R\sqrt{3}}{3}$.

5. $s'_3 = R\sqrt{3}$.

6. $s_3 = 2R\sqrt{3}$.

7. $A'_4 = 2R^2$.

8. $A_4 = 4R^2$.

9. $A'_6 = \dfrac{3R^2\sqrt{3}}{2}$.

10. $A_6 = 2R^2\sqrt{3}$.

11. $A'_3 = \dfrac{3R^2\sqrt{3}}{4}$.

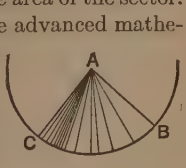
12. $A_3 = 3R^2\sqrt{3}$.

13. Find the value of $\dfrac{A'_4}{A_4}$; $\dfrac{A'_6}{A_6}$; $\dfrac{A'_3}{A_3}$; $\dfrac{s'_4}{s_4}$; $\dfrac{s'_6}{s_6}$; $\dfrac{s'_3}{s_3}$. (Use Exs. 1–12.)

14. Find the ratio of s_6 to s_3; of A_3 to A'_4.

15. Which is larger $(s_3 - s'_6)$ or $(s'_3 - s'_4)$?

366. Sector — Area of a Sector. The figure formed by an arc of a circle and the radii drawn to the ends of the arc is called *a sector*. The area of the surface inclosed within a sector is called the area of the sector. The area of a sector is best found by methods of more advanced mathematics. The nature of the proof is somewhat as follows:



The sector ABC is divided into a great many smaller parts as shown in the figure. Then triangles may be formed by connecting consecutive points

of division of the arcs. If the number of these chords is increased without bound,

the sum of the chords $\rightarrow$ the arc CB,
the sum of the triangles $\rightarrow$ the area of ABC,
the altitude of each triangle $\rightarrow$ the radius of the sector.

Then by addition and by substitution of the limits for their corresponding variables, it is shown that

$$\textit{Area of sector } ABC = \tfrac{1}{2} r \times \overset{\frown}{BC}.$$

It is also possible to prove that if the angle at the center is $n°$ that the area of the sector $= \dfrac{n°}{360°} \times \pi r^2$.

The length of the arc of a sector with central angle of $n°$ is

$$arc = \frac{n°}{360°} \times 2\pi r = \frac{n°}{180°} \times \pi r.$$

367. Corollary. *The area of a segment of a circle formed by the arc of a sector and its chord equals the area of the sector minus (or plus) the area of the triangle formed by the radii and the chord.*

$$\text{Area} = \text{Sector} \mp \triangle.$$

368. Similar Sectors—Similar Segments—Similar Arcs are those figures which have *equal-valued central angles.*

369. Corollary. *The areas of two similar sectors have the same ratio as the squares of their radii.*

Exercises. Circles—Sectors—Segments. (Numerical Calculations) (Use $\pi = 3\frac{1}{7}$.)

1. Find the circumference of a circle whose radius is 6 in.; 10 in.

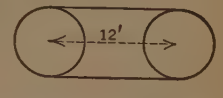
2. Find the area of a circle whose radius is 8 in.; $3\sqrt{2}$; $a\sqrt{5}$.

3. Find the circumference of a circle whose area is $144\,\pi$; $625\,\pi$.

4. Find the radius of a circle whose circumference is m inches.

5. Find the perimeter of a sector whose radius is 6 in. and central angle 60°; if the central angle is 90°; if the central angle is $m°$.

6. Find the length of an open belt which runs over two equal pulleys 18 in. in diameter and 12 ft. between their centers.



7. Find the area of the sectors of Ex. 5.

8. A regular hexagon is inscribed in a circle of radius 5 in. Find the area of a segment one boundary of which is a side of the hexagon.

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THE ALASKAN MIDDAY SUN IN WINTER

The arc was described by the sun in latitude 64 degrees 29 minutes north on December 28th. The picture was taken from the beach at Nome, Alaska, looking south across the ice fields of Bering Sea. Five exposures were made on one negative, the camera remaining in the same position for four hours. The exposures were made at 10 and 11 o'clock a.m., noon, 1 and 2 o'clock p.m. The ancients believed that the sun revolved about the earth. Can you understand why Copernicus and Galileo could not make them believe differently? What does this picture seem to prove?

9. The amount of water flowing through a pipe equals the product of the area of its cross section and the rate of flow. Using 231 cu. in. to a gallon, how much water will be emptied from a pipe in one hour if the inside diameter of the pipe is 4 in. and the rate of flow is 125 ft. a minute.

10. The radius of a wheel is 7 in. How often does the wheel revolve in covering a mile?

11. The hands of a clock are 4 in. and $2\frac{1}{2}$ in. How many inches does the tip of each hand move in 20 minutes?

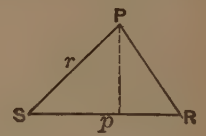
12. The perimeter of a semicircle is $41\frac{1}{7}$ millimeters. Find the area of the semicircle.

(Supplementary Exercises 12–21, p. 292, may be studied now.)

Exercises. Areas Determined by Trigonometric Functions

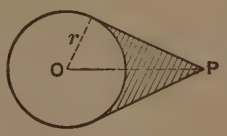
1. Find the area of a triangle if

	p	r	$\angle S$
a	12	8	70°
b	10	10	50°
c	10	10	80°



2. Find the area of the segment S if $\angle O$ is 50°; 80;° 90°.

3. Find the area of the shaded portion of the annexed figure if $r = 12$ and $\angle P = 80°$.



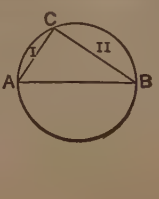
4. Find the area of a regular pentagon inscribed in a 12″ circle.

5. Find the area of a segment whose chord is a side of a regular pentagon inscribed in a 10″ circle.

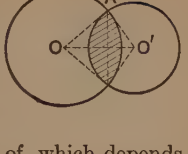
6. Find the area of a regular decagon circumscribed about a 12″ circle.

7. Find the area of the shaded portion of the annexed figure, if $p \parallel s$. Chord p is a side of a regular inscribed pentagon and chord s is a side of an inscribed equilateral triangle.

8. In the annexed figure, the diameter $AB = 20$ and $\angle B = 40°$. Find the areas of segments I and II.



9. Find the area common to the two intersecting circles, if $OA = 12$, $O'A = 8$, $\angle O = 80°$.



HINT: Find half the common chord.

10. Prove: The area of a triangle equals half the product of any two sides by the sine of their included angle.

11. Make up an exercise of your own, the solution of which depends upon the use of trigonometric functions.

SUPPLEMENTARY EXERCISES

Exercises. Regular Polygons

1. Find the radius of an equilateral triangle whose side is $4\sqrt{3}$.

2. Prove that the side of the inscribed regular hexagon is double the apothem of the inscribed equilateral triangle.

3. Prove that the side of the equilateral triangle circumscribed about a circle is double the side of the equilateral triangle inscribed in the circle.

4. If the radius of a circle is R, prove that the sides of the inscribed square and the circumscribed square are respectively equal to $R\sqrt{2}$ and $2R$.

5. If the radius of a circle is R prove that the side of the circumscribed regular hexagon equals $\frac{2R}{3}\sqrt{3}$.

6. Sketch an irregular polygon and mark the parts you must know in order to find its area. Is the formula of § 349 suitable for finding its area? Why?

7. The diagonal of a square is $6\sqrt{2}$; find the area by the formula of § 349.

8. The apothem of a regular hexagon is $\frac{a}{2}\sqrt{3}$. Find the area.

9. Find the central angles for the following regular polygons: an equilateral triangle; a square; a pentagon; a hexagon; an octagon; a decagon.

10. Prove that the line which joins the center of a regular polygon to the mid-point of a side is an apothem.

11. Find the area of the hexagram (six-pointed star) formed by producing the sides of a regular hexagon, if the radius of the hexagon is 10 in. Make a list of all the Facts needed for this exercise.

Exercises. Circles—Sectors—Segments

12. Find the area of the circle which is circumscribed about an equilateral triangle whose perimeter is 30 in.

13. Find the area of the figure bounded by the arcs DE, EF, and FD in an equilateral triangle whose side is 12 in.

14. In the annexed figure, the arcs are quadrants of a circle whose radius is r. Find the area of the shaded figure (called a lune).

15. The weight that can be supported safely by a hemp rope D inches in diameter is found by the formula, $Wt. = 1120 \times D^2$. Find the safe load on a hemp rope 3 in. in diameter.

16. Prove that the area of a circle equals four times the area of a second circle whose diameter equals the radius of the first circle.

17. The areas of two sectors of the same circle have the ratio of 2:3. Find the ratio of their central angles.

18. Prove that the area of the segment of a circle whose chord is the side of an equilateral triangle inscribed in a circle with radius R equals $\frac{R^2}{12}(4\pi - 3\sqrt{3})$.

19. Find the area of the segment of a circle whose chord is the side of a regular hexagon inscribed in a circle with radius R.

20. On the sides of an equilateral triangle whose radius is 10 in. semicircles are drawn internally. Find the area of the total figure.

SUMMARY OF CHAPTER FIVE

Regular Polygons

1. A regular polygon may be inscribed in or circumscribed about a circle.

2. Regular polygons of the same number of sides are similar.

3. The perimeters of regular polygons of the same number of sides have the same ratio as their radii or as their apothems.

4. The areas of regular polygons of the same number of sides have the same ratio as the squares of their radii or as the squares of their apothems or as the squares of their perimeters.

5. The area of a regular polygon equals half the product of the apothem by the perimeter.

Circles

1. A circle may be inscribed in or circumscribed about a regular polygon.

2. The circumferences of two circles have the same ratio as their radii or as their diameters.

3. The ratio of the circumference of a circle to its diameter equals a constant number called π.

4. The circumference of a circle equals π times the diameter.

5. The area of a circle equals π times the square of the radius.

6. The areas of two circles have the same ratio as the squares of their radii.

7. The area of a sector of a circle equals half the product of the radius by the length of its arc; or, $\frac{n}{360}$ times the area of the circle, where n is the number of degrees in its central angle.

8. The areas of similar sectors have the same ratio as the squares of their radii.

REVIEW ORAL EXERCISES. DIAGNOSTIC TEST

1. What is a regular polygon?

2. What kind of polygon may be inscribed in a circle?

3. In what kind of polygon may a circle be inscribed?

4. What line in a circumscribed polygon is the same as the radius of the circle?

5. What line in an inscribed regular polygon is the same as the radius of the circle?

6. If a regular circumscribed polygon is given, describe how a regular inscribed polygon of double the number of sides may be found.

7. If a regular inscribed polygon is given, describe how a regular circumscribed polygon of half the number of sides may be found. Can such a polygon always be found? Why?

8. If the number of sides of a regular inscribed polygon is increased without bound, what is the limit (*a*) of the perimeter? (*b*) of the apothem? (*c*) of the area? (*d*) of the angle at the center? (*e*) of the angle at a vertex? (*f*) of the radius? (*g*) of each side?

9. What is the formula for the area of a regular polygon?

10. Explain how to find the area of an irregular polygon.

11. What regular polygon has a perimeter equal to eight times its apothem?

12. What is the radius of a circle whose circumference and area have the same absolute value?

13. What is the ratio of the circumferences of the two circles circumscribed about and inscribed in the same square?

14. Is any polygon inscribed in a circle regular?

15. If the number of sides of a regular polygon is doubled, is the area doubled? Compare the sizes of the respective central angles.

16. If the radius of a circle is doubled, is the diameter doubled? Is the circumference doubled? Is the area doubled? Why?

17. If the area of a sector is doubled, is the radius doubled? Is the central angle doubled?

18. What regular polygon has its side equal to its radius?

19. What parts are needed to find the area of a segment of a circle?

20. State all the properties of the apothem of a regular polygon.

MISCELLANEOUS EXERCISES

1. Prove that an angle at the center of any regular polygon equals an exterior angle of the polygon.

2. In what regular polygon is the apothem half the side of the polygon? Why?

3. If K = area of a circle, H = area of its circumscribed hexagon, and h = area of its inscribed hexagon, find the ratio of K to $\frac{H+h}{2}$.

4. If two circles are concentric, show that the area of the ring between them equals the area of a circle having for its diameter a chord of the larger circle which is tangent to the smaller.

5. If R is the radius of a circle, show that $\frac{2}{3}R\sqrt{3}$ is the side and $2R^2\sqrt{3}$ is the area of the inscribed regular hexagon.

6. On the sides of a square whose side is s, semicircles are drawn externally. Find the total area of the inclosed figure.

7. The area of the circle circumscribed about a regular hexagon is 12π. Find the apothem.

8. The area of the circle circumscribed about an isosceles right triangle is 32π. Find the area of the triangle.

9. Prove that the altitude of an inscribed equilateral triangle is $\frac{3}{4}$ of the diameter of the circle.

10. Prove that the altitude of the circumscribed equilateral triangle equals three times the radius of the circle.

11. Prove that the diagonals which join the alternate vertices of a regular hexagon form another regular hexagon.

12. Find the radius, the apothem, and the area of the following regular polygons:

a) a square whose side is 5;	*d*) a square whose side is s;
b) a hexagon " " " 6;	*e*) a hexagon " " " s;
c) a triangle " " " 8;	*f*) a triangle " " " s;

13. The ancient Egyptians said, "Construct a square whose side is $\frac{8}{9}$ of the diameter of a circle, and its area will equal the area of the circle." Compute their value of π. (See § 361.)

14. In a bake-shop two pies are made of the same materials and are of the same thickness. An 8-inch pie sells for 20 cents and a 12-inch pie sells for 40 cents. Which is the better bargain?

15. (*a*) The manufacturer of a certain automobile designed the speedometer to be used in connection with $30 \times 3\frac{1}{2}$ tires on the wheels. The owner changed to an oversize tire 31×4. How did this effect the registering of the speedometer? What was the true speed of the car when the speedometer showed 30 mi. per hour?

(*b*) Show that changing from tires A inches in diameter across the wheel to an oversize of B inches, the speed recorded on the speedometer is $\frac{100(B - A)}{A}$ % slower than the true speed of the car.

16. The area of any regular polygon may be expressed by the formula, $A = ks^2$, in which s is a side and k is a constant factor. Find the value of k, if the number of sides of the regular polygon is 3; 4; 6; 8.

17. If a 6-inch chord of the larger of two concentric circles has an arc of 120° and is tangent to the smaller circle, find the ratio of the areas of the circles; find the ratio of their circumferences.

18. Four 24-inch wires are bent into the form of an equilateral triangle, a square, a regular hexagon, and a circle respectively. Find the area of each figure.

19. Find the ratio of the areas of the circles circumscribed about and inscribed in (*a*) an equilateral triangle; (*b*) a square; (*c*) a regular hexagon.

20. Six equal coins of radius r are placed tangent to a seventh coin of the same size. If the middle coin is removed, find the area of the inner figure.

Courtesy Keystone View Co., Inc.

THE CATHEDRAL OF MILAN

The beautiful Cathedral of Milan would not have been possible without the aid of geometry. Its forest of pinnacles, its 2300 exterior statues of marble, its magnificent stained-glass windows, and its harmonious union of geometric figures have earned for it the title of "the eighth wonder of the world." It is the product of Italian, Gallic, and Teutonic architecture.

CHAPTER SIX

INEQUALITIES—CONSTRUCTIONS—LOCI

INEQUALITIES

370. Symbols. The general symbol of inequality is $\neq$. The expression $a \neq b$ is read, "a is not equal to b." This general symbol does not tell, however, which of the quantities is the greater; two other symbols are used to show such relation. The symbol $>$ means "is greater than"; the symbol $<$ means "is less than." $A > B$ is read, "A is greater than B," and $B < A$ is read, "B is less than A." It should be observed that the opening of these inequality symbols is turned in either case towards the greater quantity. *The inequalities $A > B$ and $B < A$ are equivalent expressions;* that is, they have the same meaning. An equality $X = Y$ may be written $Y = X$; so also an inequality $A > B$ may be written $B < A$.

371. Orders. If $a > x$ and $b > y$, the pairs of quantities are unequal *in the same order*, that is, the left-hand members are the greater. If $a > x$ and $c < z$, the quantities are unequal *in the reverse order*. Only inequalities of positive quantities are considered in elementary geometry.

Oral Exercises

1. What is the meaning of the expression $p > r$? Write the expression in another order.

2. Contrast as to order the pair of inequalities $b > c$ and $d > e$ with the pair $r > s$ and $h < g$.

3. If $a > b$ and $b = x$, state a Fact in support of the inequality $a > x$.

4. If $a > b$ and $a = c$, what conclusion follows? Why?

PRINCIPLES OF INEQUALITIES

Principle 43. A magnitude may be substituted for its equal in an inequality. For example; if $a > b$ and $b = x$, $a > x$.

Principle 44. If unequals are operated on by equals, the results are unequal in the same order. For example; if $a > b$ and $c = d$,

(1) $a + c > b + d$, by addition;

(2) $a - c > b - d$, by subtraction;

(3) $ac > bd$, by multiplication;

(4) $\frac{a}{c} > \frac{b}{d}$, by division.

Principle 45. If unequals are added to unequals of the same order, the sums are unequal in the same order. For example; if $a > b$ and $x > y$, $a + x > b + y$.

Principle 46. Equal powers or equal roots of unequals produce results which are unequal and in the same order. For example; if $a > b$, $a^2 > b^2$, and $\sqrt{a} > \sqrt{b}$.

Principle 47. If unequals are subtracted from equals, the differences are unequal in the reverse order. For example, if $p = r$ and $s > t$, $p - s < r - t$.

Oral Exercises

1. Apply *Prin.* 43 to $AB > CD$ and $CD = EF$. If $m - n < p$ and $m = t$, why is $t - n < p$?

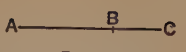
2. Apply *Prin.* 44 to the expressions $20 > 10$ and $5 = 5$.

3. If $PR = \frac{1}{2}\,PS$, $AB = \frac{1}{2}\,AC$ and $PR > AB$, what is true of PS and AC? Why?

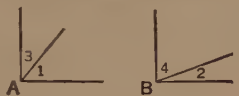
4. If $\angle 1 > \angle 3$ and $\angle 2 > \angle 4$, why is $\angle 1 + \angle 2 > \angle 3 + \angle 4$?

5. Apply *Prin.* 47. to the expressions $8 = 8$ and $5 > 2$.

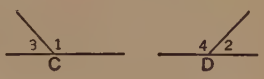
6. If $AC = PS$ and $AB > PR$, what is true of BC and RS? Why?



7. If $\angle A = \angle B = 90°$ and $\angle 1 > \angle 2$, what is true of $\angle 3$ and $\angle 4$? Why?



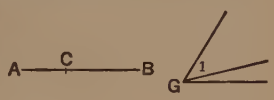
8. If $\angle C = \angle D = 180°$ and $\angle 1 > \angle 2$, what is true of $\angle 3$ and $\angle 4$? Why?



372. Theorem. *The greater of two angles has the less complement and the less supplement.*

NOTE. This is the general statement of Exercises 7 and 8 above, the proofs of which constitute the proof of this theorem.

Principle 48. A magnitude is greater than any of its parts. For example, in the annexed figures, $AB > AC$ and $\angle G > \angle 1$.



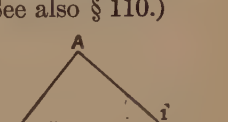
373. Corollary. *An exterior angle of a triangle is greater than either non-adjacent interior angle.* (See also § 110.)

In the annexed figure,

$\angle 1 = \angle A + \angle B$. Why?

$\angle A + \angle B > \angle A$. Why?

$\angle 1 > \angle A$. Why?



Principle 49. If three quantities are so related that the first is greater than the second and the second is greater than the third, then the first is greater than the third. For example, if $X > Y$ and $Y > Z$, then $X > Z$.

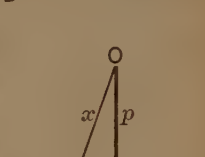
Principle 50. Two quantities must possess one of three relations to each other: they are equal, or the first is greater than the second, or the first is less than the second. For example, if A and B are two quantities, $A = B$, or $A > B$, or $A < B$.

Principle 22. A straight line is the shortest line between two points.

374. Corollary. *The sum of two sides of a triangle is greater than the third side.*

In the annexed figure,
$AB + AC > BC$. Why?

Principle 23. The perpendicular from a point to a line is the shortest line between the point and the line.



(This Principle may be proved as follows, if the Pythagorean Theorem has been proved.)

Given: p is the perpendicular from O to b, x is any other line from O to b.

To prove: $p < x$.

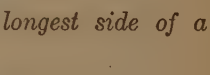
Proof:

STATEMENTS	FACTS
1. $x^2 = p^2 + b^2$.	1. Why?
2. $p^2 + b^2 > p^2$.	2. Why?
3. $x^2 > p^2$.	3. Why?
4. $x > p$ or $p < x$.	4. Why?

375. Corollary. *The hypotenuse is the longest side of a right triangle.*

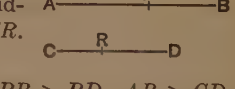
Oral Exercises. Inequalities

1. If $AB > CD$ and if P and R are the midpoints of AB and CD respectively, $AP > CR$. Why?



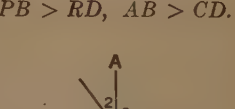
2. In the same figure, if $AP > CR$ and $PB > RD$, $AB > CD$. Why?

3. If $AB \perp CD$ and $\angle 1 > \angle 4$, $\angle 2 < \angle 3$. Why?



4. In $\triangle PRS$ if $\angle S = \angle 1$ and $\angle 1 > \angle R$, $\angle S > \angle R$.

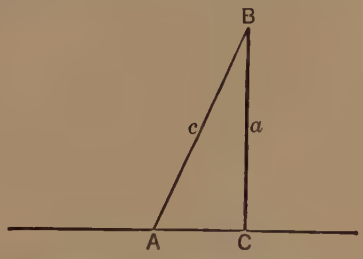
5. In $\triangle PRS$ if $\angle P > \angle 1$ and $\angle 1 > \angle R$, $\angle P > \angle R$.



6. Prove that the sum of the diagonals of a square is less than the perimeter.

Proposition 1 (C)

376. Theorem. *There is only one perpendicular to a line from a given point.*



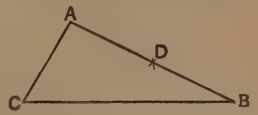
Given: $a \perp AC$ from B, a point without AC.

To prove: Any other line from B to AC (such as c) is not perpendicular to AC.

Proof:	STATEMENTS	FACTS
1.	$a \perp AC$.	1. Given.
2.	In rt.$\triangle ABC$, $\angle A + \angle B = 90°$.	2. Why?
3.	$\angle A < 90°$.	3. Why?
4.	c is not perpendicular to AC.	4. Why?

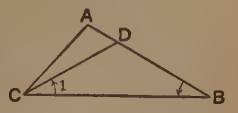
377. Equalities in Connection with Inequalities. When one line is greater than another it may be assumed (see *Prin.* 10) that there is an internal point of the greater, such that the segment bounded by the point and an extremity of the line equals the smaller.

For example, if $AB > AC$, point D may be taken on AB so that $AD = AC$.



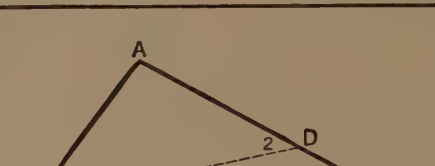
A similar assumption may be made for angles.

For example, if $\angle C > \angle B$, CD may be drawn so that $\angle 1 = \angle B$.



Proposition 2 (c n)

378. Theorem. *Angles opposite unequal sides of a triangle are unequal and the angle opposite the greater side is the greater.*



Given: $AB > AC$ in $\triangle ABC$.

To prove: $\angle C > \angle B$.

HINT: Use § 377 and § 373.

Proof:

STATEMENTS	FACTS
1. $AB > AC$.	1. Why?
2. Let $AD = AC$.	2. § 377.
3. Draw CD.	3. *Prin.* 13.
4. $\angle C > \angle 1$.	4. *Prin.* 48.
5. $\angle 1 = \angle 2$.	5. § 79.
6. $\angle 2 > \angle B$.	6. § 373.
7. $\angle C > \angle 2 > \angle B$.	7. *Prin.* 43.
8. $\angle C > \angle B$.	8. *Prin.* 49.

379. Opposite Theorems. If the hypothesis and the conclusion of a theorem are changed to the *negative* form, the theorem thus obtained is called the *opposite theorem.* § 378 is the opposite of § 79.

The relation between a *direct* theorem, the *converse* theorem, and the *opposite* theorem is shown in the following statements:

(1) **Direct Theorem.** *If two sides of a triangle are equal, the opposite angles are equal.*

(2) **Converse Theorem.** *If two angles of a triangle are equal, the opposite sides are equal.*

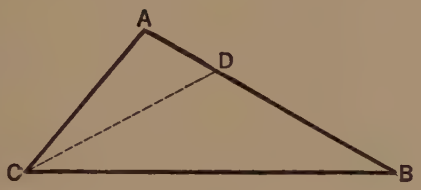
(3) **Opposite Theorem.** *If two sides of a triangle are unequal, the opposite angles are unequal.*

(4) **Opposite Converse Theorem.** *If two angles of a triangle are unequal, the opposite sides are unequal.*

Statements 1 and 2 and Statements 3 and 4 in §379 are often called *dual* or *reciprocal* theorems; the terms *sides* and *angles* of the one correspond to *angles* and *sides* of the other.

Proposition 3 (c)

380. Theorem. *Sides opposite unequal angles of a triangle are unequal and the side opposite the greater angle is the greater.*



Given: $\angle C > \angle B$ in $\triangle ABC$.

To prove: $AB > AC$.

HINT: Use § 377 and § 374.

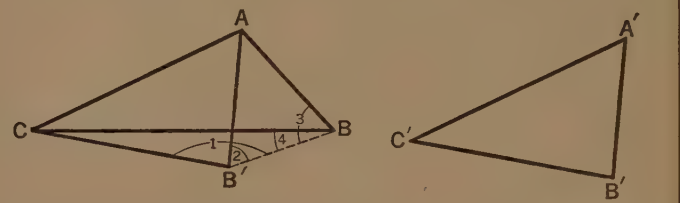
Proof:

STATEMENTS	FACTS
1. $\angle C > \angle B$.	1. Given.
2. Let $\angle BCD = \angle B$.	2. § 377.
3. $DC = DB$.	3. Why?
4. $AD + DC > AC$.	4. Why?
5. $AD + DB > AC$, or $AB > AC$.	5. *Prin.* 43.

Questions: What would be the relation of $\angle B$ and $\angle C$ in the above figure if it is assumed that $AB = AC$? What would be their relation if it is assumed that $AB < AC$? Do the previous assumptions lead to statements which are contrary to fact? What is this Fact? Where is it stated?

Proposition 4 (c n)

381. Theorem. *In triangles with two sides of one equal respectively to two sides of another, but the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.*



Given: $AC = A'C'$, $AB = A'B'$, $\angle A > \angle A'$ in the $\triangle$.

To prove: $BC > B'C'$.

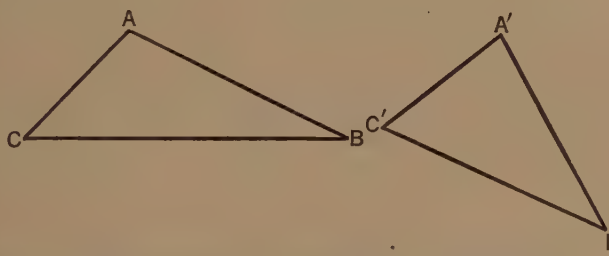
HINT: Use § 380 and *Prin.* 48.

Proof:

STATEMENTS	FACTS
1. Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $A'C'$ coincides with its equal AC.	1. $A'C' = AC$, given.
2. $A'B'$ will fall within $\angle A$.	2. $\angle A > \angle A'$, given.
3. Draw BB'.	3. *Prin.* 13.
4. $\angle 1 > \angle 2$.	4. Why?
5. $AB = AB'$.	5. Why?
6. $\angle 2 = \angle 3$.	6. Why?
7. $\angle 1 > \angle 3$.	7. Why?
8. $\angle 3 > \angle 4$.	8. Why?
9. $\angle 1 > \angle 4$.	9. *Prin.* 49.
10. $BC > B'C$, that is, $BC > B'C'$.	10. § 380.

Proposition 5 (c n)

382. Theorem. *In triangles with two sides of one equal respectively to two sides of another, but the third side of the first greater than the third side of the second, the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*



Given: $AC = A'C'$, $AB = A'B'$, $BC > B'C'$ in the $\triangle$.

To prove: $\angle A > \angle A'$.

HINT: Use *Prin.* 50.

Proof: STATEMENTS	FACTS
1. $\angle A = \angle A'$, $\angle A < \angle A'$, or $\angle A > \angle A'$.	1. *Prin.* 50.
2. $AC = A'C'$ and $AB = A'B'$.	2. Given.
3. If $\angle A = \angle A'$, $\triangle ABC \cong \triangle A'B'C'$ and $BC = B'C'$.	3. § 65 and §77.
4. This is impossible.	4. $BC > B'C'$. Given.
5. If $\angle A < \angle A'$, $BC < B'C'$.	5. § 381.
6. This is impossible.	6. Fact 4.
7. $\angle A > \angle A'$.	7. *Prin.* 50.

383. Indirect Methods of Proof. There are three methods of indirect proof used in geometry. They are:

(1) The *Method of Coincidence* (§ 100) which consists of showing that a given line coincides with another line which answers the required conditions.

(2) The *Reductio ad Absurdum Method* (§ 115) which consists of showing that the negative of a theorem leads to a condition which is contrary to fact.

(3) The *Exclusion Method* which consists of showing that two of three possible conditions of relation between two quantities are impossible. This is the method used in proving § 382.

Many times Converse Theorems and Inequalities are most easily established by either of the last two forms of indirect proof.

(A group of exercises to be proved by indirect methods is found on p. 308.)

Exercises. Inequalities in Polygons

1. An obtuse angle of a triangle is greater than either of the other two angles.

2. If the angle at the vertex of an isosceles triangle is obtuse, the base is the longest side.

3. If the alternate vertices of any hexagon are connected by lines to form a triangle, the perimeter of the hexagon is greater than the perimeter of the triangle.

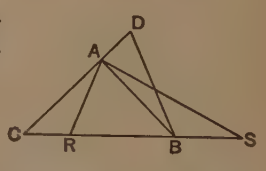
4. The altitude drawn to any side of a triangle is less than half the sum of the other sides.

5. In the annexed figure $AB = AC$.

a) Prove $CD > BD$.

b) Prove $AS > AC$.

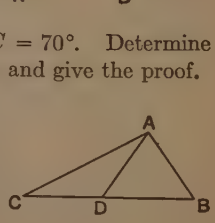
c) Prove $AC > AR$.



6. In $\triangle ABC$, $\angle A = 50°$, $\angle B = 60°$, and $\angle C = 70°$. Determine which side is the longest and which is the shortest and give the proof.

7. In $\triangle ABC$, $\angle B > \angle C > \angle A$. Prove that AC is the longest and BC the shortest side of the triangle.

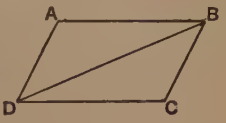
8. In $\triangle ABC$, median AD is drawn forming an acute angle ADB. Prove $AC > AB$.



9. If $AC > AB$ in $\triangle ABC$ and median AD is drawn, prove $\angle ADC$ obtuse.

10. If AC and BD are diagonals of a quadrilateral, prove $AC + BD > AB + CD$.

11. In the annexed figure, $ABCD$ is a parallelogram with $\angle A > 90°$. Prove that diagonal BD is greater than any side of the parallelogram.

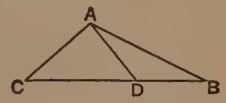


12. In the figure of Ex. 11, as $\angle A$ decreases prove that diagonal BD also decreases, provided the lengths of the sides of the parallelogram remain the same.

13. The sum of the diagonals of any quadrilateral is less than the perimeter, but greater than half the perimeter.

14. In the annexed figure, show that $AB + BC > AD + DC$.

15. The sum of the three altitudes of a triangle is less than the perimeter.

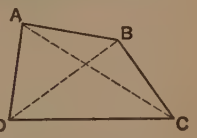


16. If side AB of an equilateral triangle ABC is produced to D and CD is drawn, prove $AD > CD > AC$.

17. In the annexed figure, $BC = AD$ and $\angle D > \angle C$.

a) Prove diagonal $AC >$ diagonal BD.

b) Prove $\angle B > \angle A$.

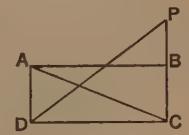


18. If the diagonals of a parallelogram are unequal, the figure cannot be a rectangle.

HINT: Use § 382.

19. In the annexed figure, $ABCD$ is a rectangle. Prove $PD > AC$.

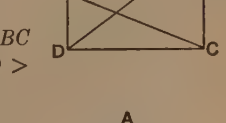
HINT: $PC^2 + DC^2 > AD^2 + DC^2$.



20. If the leg AB of an isosceles triangle ABC is produced below the base to D, prove $\angle ACD > \angle ADC$.

21. In the annexed figure, $AC < BC$ and $AD = BE$. Prove $AE < BD$.

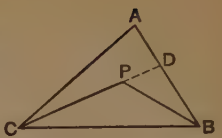
HINT: Use § 381.



22. The sum of the lines from any point within a triangle to the three vertices is greater than one-half the sum of the sides.

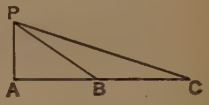
23. If from point P within triangle ABC lines PB and PC are drawn, prove $AB + AC > PB + PC$.

HINT: Draw PD.



24. Of two lines drawn from a point in a perpendicular to a given line, cutting off on the given line unequal segments from the foot of the perpendicular, the more remote is the greater.

25. Of two unequal lines from a point to a line, the greater cuts off the greater segment from the foot of the perpendicular drawn from the point to the line.



NOTE.—The Pythagorean Theorem (if it has been previously studied) when applied to Ex. 24 or Ex. 25 gives an interesting form of proof.

Exercises. Indirect Proofs

1. Prove that there can be only one mid-point of a line.

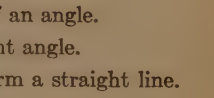
HINT: Assume that P and P' are mid-points of XY.

2. Prove that there can be only one bisector of an angle.

3. Prove that a triangle can have only one right angle.

4. The bisectors of a pair of vertical angles form a straight line.

5. In the figure, D and E are the mid-points of AC and AB. Prove that BD and CE do not bisect each other.



6. Only one perpendicular can be drawn to a line at a given point in that line. Prove.

7. Only one perpendicular can be drawn to a line from a given point.

8. If an angle of a triangle is obtuse, the other angles are acute.

9. If an angle of an isosceles triangle is obtuse, it is the vertex angle.

10. Only two equal straight lines can be drawn from a point to a line.

11. Prove Proposition 3 (§ 380) by the third indirect method of § 383.

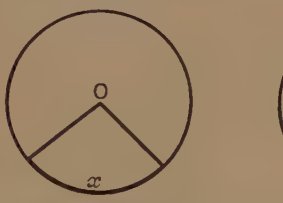
12. Prove that the bisector of the vertex angle of an isosceles triangle bisects the base by the second indirect method of § 383.

13. If $ax = b$, prove by the Exclusion Method that $x = \frac{b}{a}$.

14. Prove that two right angles are equal.

Proposition 6

384. Theorem. *The greater central angle of a circle (or of equal circles) has the greater arc; conversely, the greater arc has the greater central angle.*



Given: $\angle O > \angle O'$ with $\overset{\frown}{x}$ and $\overset{\frown}{x'}$ in equal circles.

To prove: $\overset{\frown}{x} > \overset{\frown}{x'}$.

HINT: Use *Prin.* 31 and *Prin.* 43.

Proof: STATEMENTS	FACTS
1. Circle O = circle O'.	1. Given.
2. $\angle O$ = numerically $\overset{\frown}{x}$ and $\angle O'$ = numerically $\overset{\frown}{x'}$.	2. Why?
3. $\angle O > \angle O'$.	3. Why?
4. $\overset{\frown}{x} > \overset{\frown}{x'}$.	4. Why?

(The converse proof is left to the pupil.)

NOTE: The angles referred to are each less than 180°.

Oral Exercises

1. If a central angle is increased is its arc increased? Why? If the angle is doubled is the arc doubled? Why?

2. Is the theorem of § 384 true of central angles and arcs of unequal circles?

3. Does an arc degree of longitude upon the earth's surface vary in different places? Why? Do degrees of latitude vary? Why?

Proposition 7 (c n)

385. Theorem. *The greater arc of a circle (or of equal circles) has the greater chord; conversely, the greater chord has the greater arc.*

Given: $\overset{\frown}{x} > \overset{\frown}{x'}$ with chords c and c' in equal circles.

To prove: $c > c'$.

HINT: Use § 381.

Proof:

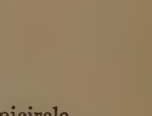
STATEMENTS	FACTS
1. Circle O = circle O'.	1. Why?
2. Draw radii a, b, a' and b'.	2. *Prin.* 13.
3. $\overset{\frown}{x}$ = numerically $\angle O$ and $\overset{\frown}{x'}$ = numerically $\angle O'$.	3. Why?
4. $\overset{\frown}{x} > \overset{\frown}{x'}$.	4. Why?
5. $\angle O > \angle O'$.	5. *Prin.* 43.
In $\triangle abc$ and $\triangle a'b'c'$,	
6. $a = a'$ and $b = b'$.	6. *Prin.* 33.
7. $c > c'$.	7. § 381.

(The converse proof is left to the pupil.)

NOTE: The arcs referred to are each less than a semicircle.

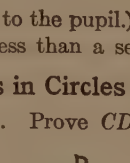
Exercises. Inequalities in Circles

1. In the annexed figure, $BD > AC$. Prove $CD > AB$.

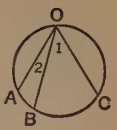


2. State and prove the converse of Ex. 1.

3. In the annexed figure, $\angle B = 40°$ and $\angle C = 60°$. Which is the largest arc? Prove your answer.



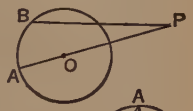
4. In the annexed figure, $OA = OC$ and $\angle 1 > \angle 2$. Prove arc $BCO >$ arc BAO.



5. If $\triangle PRS$ is inscribed in a circle and $\angle P > \angle R$, prove $\overset{\frown}{RS} > \overset{\frown}{PS}$.

6. State and prove the converse of Ex. 5.

7. The longest secant that can be drawn to a circle from an external point passes through the center. (Prove $PA > PB$.)



8. In the annexed figure, AC is a diameter and $\angle 1 > \angle 2$. Prove $AB > AD$.

9. In the annexed figure, $\angle 1 > \angle 2$. Prove $\overset{\frown}{x} > \overset{\frown}{y}$. State as a general theorem.

Courtesy Marion Steam Shovel Co.

A ONE-MAN SHOVEL

This picture shows one of the largest "one-man shovels" in the world. It takes eight dumpcart-loads at one bite and can take a bite a minute. Can you tell which theorems of Inequalities are illustrated by the operation of this machine? How many triangles change the lengths of their sides and the sizes of their angles?

Proposition 8 (c n)

386. Theorem. *The greater of two chords of a circle (or of equal circles) is nearer the center; conversely, the chord nearer the center is the greater.*

Given: Circle O = circle O', $d \perp c$, $d' \perp c'$, $c > c'$.

To prove: $d < d'$.

HINT: Draw radii and apply the Pythagorean Theorem.

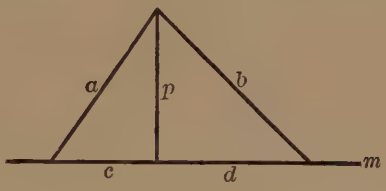
Proof: STATEMENTS	FACTS
1. Draw radii r and r'.	1. *Prin.* 13.
2. $d \perp c$ and $d' \perp c'$.	2. Why?
3. $b = \frac{1}{2} c$ and $b' = \frac{1}{2} c'$.	3. § 200.
4. $c > c'$.	4. Given.
5. $b > b'$.	5. Why?
6. $b^2 > b'^2$.	6. *Prin.* 46.
7. Circle O = circle O'.	7. Why?
8. $r = r'$.	8. Why?
9. $r^2 = r'^2$.	9. Why?
10. $r^2 - b^2 < r'^2 - b'^2$.	10. *Prin.* 47.
11. $r^2 - b^2 = d^2$ and $r'^2 - b'^2 = d'^2$.	11. Why?
12. $d^2 < d'^2$.	12. Sub.
13. $d < d'$.	13. *Prin.* 46.

(The converse proof is left to the pupil.)

387. Corollary. *A diameter of a circle is the greatest chord.*

Proposition 9

388. Theorem. *Of two lines drawn to a line from any point in a perpendicular to that line, the one with the greater projection upon the given line is the greater; and conversely.*



Given: $p \perp m$; b and a from a pt. in p make $d > c$ in m.

To prove: $b > a$.

HINT: Use the Pythagorean Theorem and the Substitution Method.

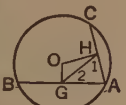
Proof: STATEMENTS	FACTS
1. $b^2 = d^2 + p^2$ and $a^2 = c^2 + p^2$.	1. Pythagorean Theorem.
2. $d > c$.	2. Given.
3. $d^2 > c^2$.	3. *Prin.* 46.
4. $p^2 = p^2$.	4. Iden.
5. $d^2 + p^2 > c^2 + p^2$.	5. *Prin.* 44.
6. $b^2 > a^2$.	6. *Prin.* 43.
7. $b > a$.	7. *Prin.* 46.

(The converse proof is left to the pupil.)

Exercises. Inequalities in Circles

1. In the annexed figure, GH and KL are both perpendicular to radius OS. Prove $GH > KL$.

2. The shortest chord through a point within a circle is perpendicular to the radius through that point.



3. If OG and OH are perpendiculars from the center and $\angle 1 > \angle 2$, prove $AB > AC$.

SUMMARY OF INEQUALITIES

Lines are unequal

(1) if one is a straight line and the other is a broken line or a curved line between two points;

(2) if they are the hypotenuse and a leg of a right triangle;

(3) if they are sides of a triangle opposite unequal angles;

(4) if they are sides of two triangles with the other sides respectively equal, but the included angles unequal;

(5) if they are chords of a circle (or of equal circles) with unequal arcs;

(6) if they are chords of a circle (or of equal circles) unequally distant from the center.

Angles are unequal

(1) if they are an exterior angle and a non-adjacent interior angle of a triangle;

(2) if their complements or their supplements are unequal;

(3) if they lie opposite unequal sides of a triangle;

(4) if they are angles of two triangles with the including sides respectively equal, but the sides opposite them unequal.

Arcs of a circle (or of equal circles) **are unequal**

(1) if their central angles are unequal;

(2) if their chords are unequal.

CONSTRUCTIONS

In our course thus far we have assumed that it is possible to draw certain figures. In the following sections we shall learn how to make certain constructions and to prove that they are correct.

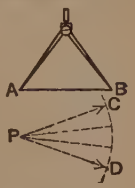
389. Instruments Used in Constructions. The only instruments permitted in the construction problems of plane geometry are the straightedge, or ruler, and the compasses. The ruler is used only for drawing straight lines; not for measuring them. The compasses are used for drawing circles and arcs and also for measuring distances.

(*Principles* 10, 11, 13, and 32 are fundamental in constructions.)

390. Preliminary Constructions.

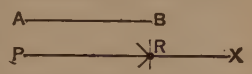
Prelim. Cons. 1. To find one or more points at a given distance from a given point.

Let AB be the given distance. Adjust the points of the compasses so that they coincide with A and B, the extremities of the line. With the given point P as a center and with a radius equal to AB draw an arc of any length. Any points upon the arc, such C and D, are at the given distance from P. Why?



Prelim. Cons. 2. To find a point on a line at a given distance from an extremity of the line.

Suppose PX is a line on which we wish to find a point whose distance from P equals AB. Adjust the compasses to coincide with A and B. With one leg of the compasses upon P as a center draw an arc intersecting PX at R. Then R is a point such that $PR = AB$. Why?

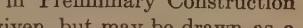


Prelim. Cons. 3. To find a line equal to a given line.

The same construction is followed as in Preliminary Construction No. 2 with the exception that PX is not given, but may be drawn as a general line.

Prelim. Cons. 4. To find the sum of two given lines.

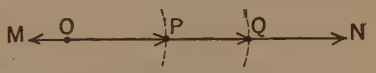
Given: Lines AB and CD



Required: Line $x = AB + CD$.

Construction: With a straightedge draw a general straight line such as

Place one leg of the compasses upon A and the other upon B. Next place one leg at any point O on MN and cut off segment $OP = AB$. In like manner, measure CD and lay off $PQ = CD$. Then line $x = OP + PQ = AB + CD$. Why?



Prelim. Cons. 5. To find n times a given line.

(Let the pupil take $n = 3$ and complete.)

Prelim. Cons. 6. To find the difference of two given lines.

(To be completed by the pupil.)

Prelim. Cons. 7. To find two points on a line which are equidistant from a given point on the line.

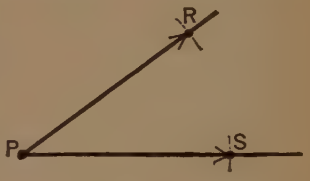
Given: Point P on line MN.

Required: Two points (R and S) on MN such that $PR = PS$.

Construction: Since PR and PS need not equal a given length, we may use any radius that is convenient in locating the points. The length of such a radius will be governed by the size of the drawing. With P as a center and with any convenient radius draw arcs intersecting MN at R and S. Then $PR = PS$. Why?

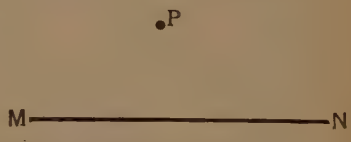
Prelim. Cons. 8. To find a point on each side of a given angle such that the points are equidistant from the vertex.

(The construction should be described by the pupil.)



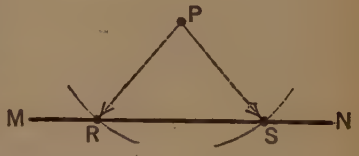
Prelim. Cons. 9. To find two points on a line which are equidistant from a point not on the line.

Given: Line MN and external point P.



Required: Two points (R and S) on MN so that $PR = PS$.

Construction: With P as a center and with any convenient radius greater than the perpendicular from P to MN, draw arcs intersecting MN at R and S. Then $PR = PS$. Why?



Prelim. Cons. 10. To find a point equidistant from two other points.

Given: Points A and B.

Required: A point (C) so that $AC = BC$.

Construction: With A and B as centers and with any convenient radius greater than $\frac{1}{2}\ AB$ draw arcs which intersect at C. $AC = BC$. Why? Why is it necessary to use a radius greater than $\frac{1}{2}\ AB$?

Oral Exercises. Prelim. Cons. 10

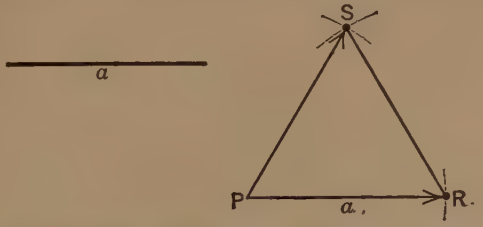
1. How many points like C may be found equidistant from A and B?

2. What may be said of the straight line joining any two points equidistant from A and B?

3. What kind of triangle is formed by joining points A, B, and C?

4. If the radius is taken equal to AB, what kind of triangle is formed?

Prelim. Cons. 11. To find an equilateral triangle whose side equals a given line.



(The construction should be described by the pupil.)

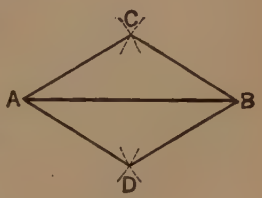
How many degrees in each angle? Why?

Prelim. Cons. 12. To find an angle of 60°.

(To be done by the pupil.)

Prelim. Cons. 13. To find an isosceles triangle if its base and one of the equal sides are given.

(To be done by the pupil.)



Prelim. Cons. 14. To find two isosceles triangles having a given line as their common base.

(To be done by the pupil.)

Oral Exercises. Prelim. Cons. 14

1. In the preceding construction, what may be said of points C and D with reference to points A and B?

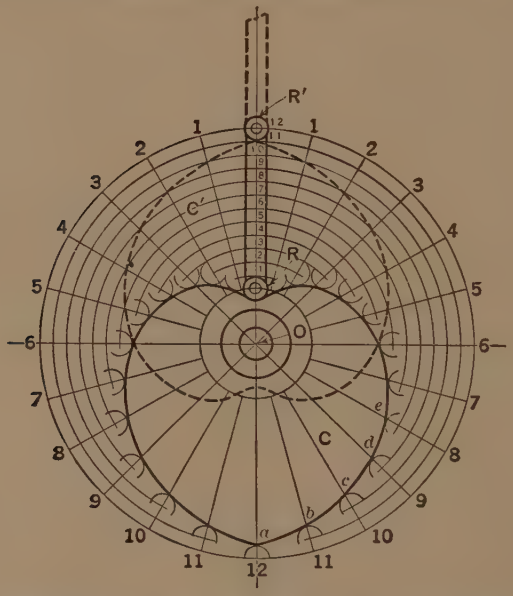
2. If CD is drawn, what is its relation to AB?

3. May points C and D be on the same side of AB? What is the advantage of constructing the figure as shown above?

4. If the same radius is used throughout in Preliminary Construction 14, what is the figure $ACBD$? What is true of the diagonals?

5. What is the relation of CD to $\angle C$?

391. Problem. A problem is a question to be solved. In geometry, there are problems of computation and problems of construction; for example, the first may ask us to **compute** the complement of an angle of 30°; the second, to **construct** the complement of an angle of 30°.



The heart-shaped cam shown here is the result of skilful construction. The points *a, b, c, d,* etc., are located at the intersection of arcs of circles (whose centers are on circles concentric at *O*) and equally spaced radii. *R* is a following roller which travels about the edge of cam *C*. *R'* and *C'* show the positions after the cam has rotated through 180°. This cam is used to translate rotating motion into reciprocating motion.

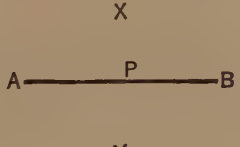
392. Parts of a Written Solution of a Construction. Every complete solution of a problem of construction, when written in formal manner, comprises the following six parts:

(1) Statement of the *Problem.*
(2) Representation of the *given* parts.
(3) Statement of what is *required.*
(4) Description of the *construction.*
(5) *Check* or proof that the construction is correct.
(6) *Discussion* of any special limitations of the problem.

NOTE: Construction lines should be visible in the completed figure.

Proposition 1 (N)

393. Problem. *To find the perpendicular bisector of a given line.*

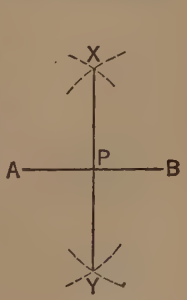


Given: Line AB.

Required: A line $XPY \perp AB$ with $AP = PB$.

HINT: What proposition in Chapter One relates to two points which determine the perpendicular bisector of a line? What Preliminary Construction shows how to locate two such points?

Construction: With A and B as centers and with any convenient radius draw arcs intersecting at X; again with A and B as centers and with any convenient radius draw arcs intersecting at Y. Draw XY intersecting AB at P. Then $XPY \perp AB$ and $AP = PB$.



Check: $AX = BX$ and $AY = BY$. Why?
$XPY \perp AB$ and $AP = PB$. Why?

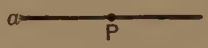
394. Corollary (C N). *To find the bisector of a given line.*

May the same construction be given as above? How does the check differ?

Proposition 2 (C N)

395. Problem. *To find a perpendicular to a given line through a given point.*

C



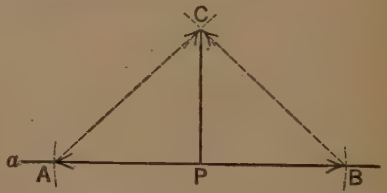
PART I. *When the point is on the line.*

Given: Line a with point P upon the line.

Required: A line such as $CP \perp a$.

HINT: How may the previous problem be adapted to this case? Is it possible to find points A and B equidistant from P and then proceed as before? Which Preliminary Constructions may be used?

Construction: With P as a center and any convenient radius describe arcs intersecting line a at A and B. With A and B as centers and with a greater radius describe arcs intersecting at C. Draw CP. Then $CP \perp AB$.



Check: (The check is left to the pupil.)

PART II. *When the point is not on the line.*

.P

Given: Line a and P an external point.

a

Required: A line such as $PY \perp a$.

Y

HINT: First apply Preliminary Construction No. 9 to find two points (A and B) on the line which are equidistant from P. Then apply Preliminary Construction No. 10 to find a point (Y) equidistant from A and B.

(Construction and check are left to the pupil.)

Construction Exercises. Perpendicular Lines

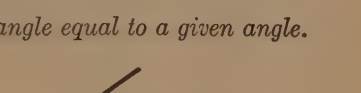
1. To find an angle which is the complement of a given acute angle.
2. To find a square whose side equals a given line.
3. To find an altitude of a given triangle.
4. To find a 45°–45°–90° triangle.
5. To find a right triangle with one leg double the other.
6. To find a 30°–60°–90° triangle.

Proposition 3 (C N)

396. Problem. *To find an angle equal to a given angle.*

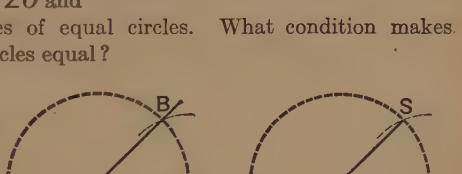
Given: Angle O.

Required: $\angle P = \angle O$.



HINT: If equal circles are drawn with O and P as centers, $\angle O$ and $\angle P$ will be central angles of equal circles. What condition makes central angles of equal circles equal?

Construction: With O and P as centers and with a convenient radius draw circles O and P. Let circle O intersect the sides of $\angle O$ at A and B. Measure chord AB and from any point R on circle P draw RS equal to AB. Draw PR and PS, $\angle P = \angle O$.



Check:

1. Circle O = circle P.	1. Why?
2. $AB = RS$.	2. Why?
3. $\overset{\frown}{AB} = \overset{\frown}{RS}$.	3. Why?
4. $\angle O = \angle P$.	4. Why?

NOTE: In the above construction it is not necessary to draw the entire circles.

Proposition 4 (C N)

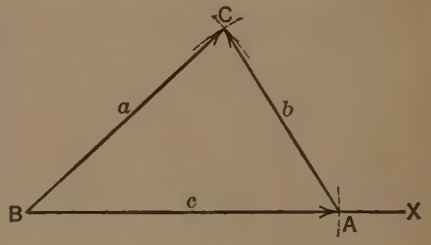
397. Problem. *To find a triangle when its three sides are given.*

Given: Lines a, b, and c.

Required: A triangle whose sides are a, b, and c.

HINT: Follow a method similar to Preliminary Construction No. 11.

Construction: Upon a base line BX draw $BA = c$. With B and A as centers and with radii equal to a and b respectively, describe arcs intersecting at C. Draw AC and BC. $\triangle ABC$ is the required triangle.

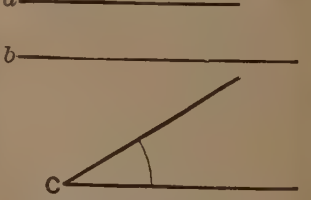


Check: (The check is left to the pupil.)

Discussion: (See § 374.)

Proposition 5 (C N)

398. Problem. *To find a triangle when two sides and the included angle are given.*

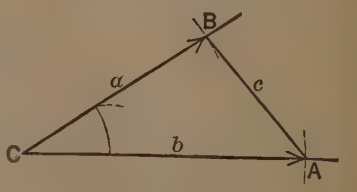


Given: a and b, two sides of a triangle, and C, their included angle.

Required: A triangle with the given parts.

HINT: Is it possible to construct an angle equal to $\angle C$? May segments equal to a and b be measured from the vertex upon the sides of the angle?

(Construction and check are left to the pupil.)

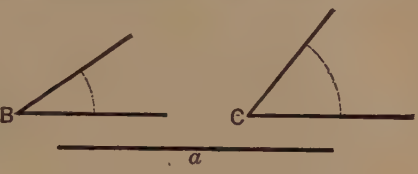


Proposition 6 (C N)

399. Problem. *To find a triangle when a side and two angles are given.*

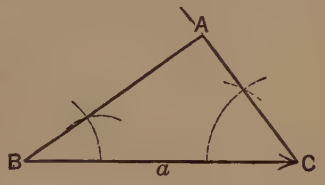
Given: Side a and angles B and C.

Required: A triangle with side a and angles B and C.



PART I. When a is the side included between the two angles.

Construction: Draw $BC = a$. At B and C construct angles equal to B and C. Let their sides intersect at A. $\triangle ABC$ is the required triangle.

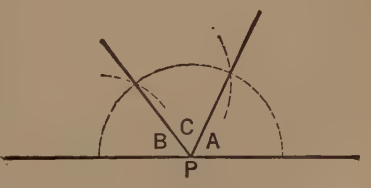


Check: (The check is left to the pupil.)

PART II. When a is not the included side.

Construction: At point P on one side of a straight line construct $\angle B$ and adjacent to that angle construct $\angle C$. $\angle A$ is the third angle of the required triangle. Why?

With a as the side included between $\angle A$ and either $\angle B$ or $\angle C$ construct the triangle as in Part I.

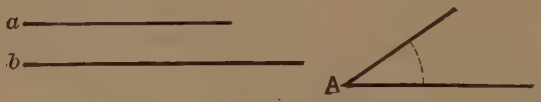


Check: (The check is left to the pupil.)

Discussion: (Tell the limitations on the size of angles B and C.)

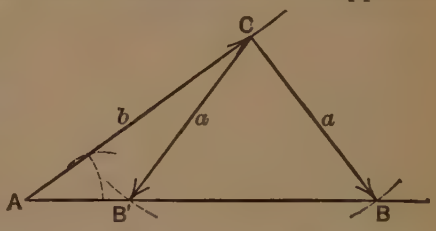
Proposition 7

400. Problem. *To find a triangle when two sides and the angle opposite one of them are given.*



Given: Sides a and b and angle A.

Required: A triangle with a and b as sides and $\angle A$ opposite side a.



Construction: Upon a side of $\angle A$ make $AC = b$. With C as a center and a radius equal to a, describe arcs cutting the opposite side at B and B'. Draw CB and CB'. Then $\triangle ABC$ and $\triangle AB'C$ each meet the given conditions.

Check: (The check is left to the pupil.)

Discussion: If the arc BB' intersects the base line at two points, there are two solutions; if the arc is tangent to the base line, there is one solution; if the arc does not intersect the base line, there is no solution.

Construction Exercises. Triangles

To find a right triangle when given

1. A leg and its adjacent acute angle.
2. A leg and its opposite acute angle.
3. The hypotenuse and an acute angle.
4. The hypotenuse and a leg.

To find an isosceles triangle when given

5. The base and a base angle.

6. One of the equal sides and the angle at the vertex.

7. One of the equal sides and a base angle.

8. The altitude to the base and the base.

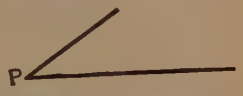
9. The altitude to one of the equal sides and the base.

Proposition 8 (C N)

401. Problem. *To find the bisector of an angle.*

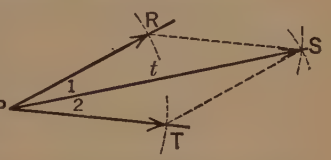
Given: Angle P.

Required: A bisector of $\angle P$.



HINT: What method is most commonly used in Chapter One to show that two angles are equal? Use Preliminary Constructions 8 and 10.

Construction: With P as a center and with any convenient radius measure PR and PT upon the sides of the given angle. With R and T as centers describe arcs intersecting at S. Draw PS or t, which is the required bisector.



Check: (The check is left to the pupil.)

Construction Exercises. Angles

1. To find a 45° angle.

2. To find a 30° angle.

3. To find a 75° angle.

4. To find an angle of a regular hexagon.

5. To find an angle of a regular octagon.

6. To find the fourth part of a given angle.

7. To find an angle equal to twice a given angle.

8. To trisect a right angle.

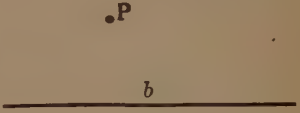
9. To find half the sum of two angles.

10. To find a base angle of an isosceles triangle when the angle at the vertex is given.

11. To find the angle at the vertex of an isosceles triangle when a base angle is given.

Proposition 9 (C N)

402. Problem. *To find a line through a given point and parallel to a given line.*

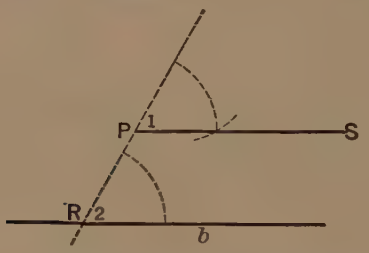


Given: Line b and point P.

Required: A line through P and parallel to b.

HINT: What is a common method of showing that two lines are parallel? Use § 396.

Construction: Through point P draw any line intersecting line b at R. Construct $\angle 1 = \angle 2$. Then $PS \parallel b$.



Check: (The check is left to the pupil.)

Discussion: The construction is impossible when point P is on the line or the line produced.

Construction Exercises. Parallelograms

1. To find a parallelogram when two adjacent sides and their included angle are given.

2. To find a parallelogram when two adjacent sides and the diagonal joining their extremities are given.

3. To find a rhombus whose sides equal a given line and one of whose angles is 60°.

4. To find a square whose perimeter is given.

5. To find a rhombus whose diagonals are given.

6. To find a parallelogram when two adjacent sides and the altitude upon one of them are given.

7. To find a square whose diagonal is given.

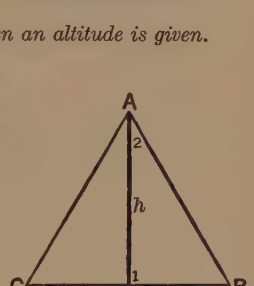
8. To find a parallelogram when one side and the diagonals are given.

9. To find a parallelogram when the diagonals and an angle included by them are given.

403. Analysis of a Problem. An analysis of a problem is the course of reasoning by which its solution is determined. In some problems the construction may be obvious; in others it may be discovered best by following a definite procedure as shown in the following exercise.

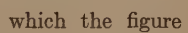
Problem: *To find an equilateral triangle when an altitude is given.*

Analysis: Suppose ABC is the required triangle whose altitude h equals the given line. Draw the altitude as a **heavy black line** and study its relation to the rest of the figure. Since h is an altitude, what is $\angle 1$? How many degrees are there in $\angle 1$? What is the relation of $\angle 2$ to $\angle A$? How many degrees are there in $\angle 2$? If h, $\angle 1$, and $\angle 2$ are known, $\triangle ADB$ may be constructed. By what proposition?



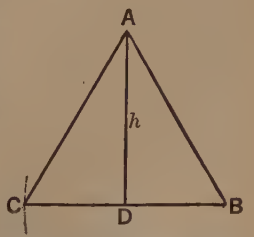
The analysis thus far gives a *basic triangle* from which the figure may be completed.

Given: A line h.



Required: An equilateral triangle with altitude equal to h.

Construction: By means of Part I of § 399 and using h, 30°, and 90° as the parts, construct the basic triangle ADB. Produce BD so that $DC = DB$. Draw AC. Then $\triangle ABC$ is the required equilateral triangle with altitude h.



Check: (The check is left to the pupil.)

404. Steps of an Analysis.

1. Make a sketch showing how the figure would appear when completed. (Such a sketch corresponds to an architect's blueprint which shows in advance the finished structure.)

2. Draw all the given parts as heavy lines, or use check marks.

3. Find a part of the figure that is fully determined. This is usually a triangle and is called the basic triangle.

4. From the basic triangle try to determine successively other points, lines, or angles until the figure is completed.

NOTE: In some cases it may be necessary to draw additional lines in order to obtain a basic triangle.

Construction Exercises

To find an isosceles triangle when given

1. The altitude upon the base and one of the equal sides.

2. The altitude upon the base and the angle at the vertex.

3. The base and the median to the base.

4. One of the equal angles and the median to the base.

To find a square when given

5. The radius of the inscribed circle.

6. The radius of the circumscribed circle.

To find a rhombus when given

7. An angle and the diagonal from its vertex.

8. A diagonal and an angle opposite it.

9. A side and a diagonal.

10. A side and the angle it forms with a diagonal.

Proposition 10

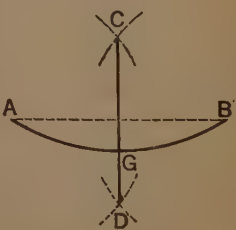
405. Problem. *To find the bisector of a given arc.*

Given: Arc AB.

Required: To bisect $\overset{\frown}{AB}$.

HINT: What proposition in Chapter Two relates to the bisector of an arc?

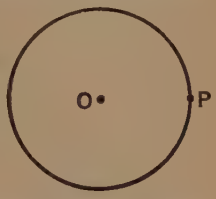
Construction: Draw chord AB. Find CD, the perpendicular bisector of chord AB. Then $\overset{\frown}{AG} = \overset{\frown}{GB}$.



Check: (The check is left to the pupil.)

Proposition 11 (C N)

406. Problem. *To find a tangent to a circle through a point on the circle.*

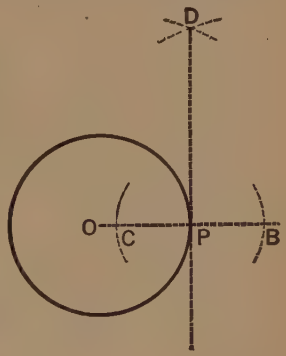


Given: Circle O and point P on the circle.

Required: A tangent to the circle through P.

HINT: Recall the theorems in connection with a tangent. What is the relation between a tangent and a radius drawn to the point of tangency?

Construction: Draw radius OP. Produce OP any convenient length to B. Find $PD \perp OB$. Then PD is tangent to the circle at P.



Check: (The check is left to the pupil.)

Proposition 12 (C N)

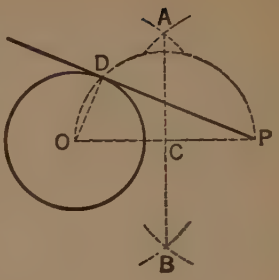
407. Problem. *To find a tangent to a circle through a point without the circle.*

Given: Circle O and point P without the circle.

Required: A tangent to the circle through P.

HINT: What is the theorem concerning the relation of a radius to a tangent? What point must be found on the circle?

Construction: Draw OP. Find ACB the bisector of OP. With C as a center and radius OC draw semicircle ODP intersecting the circle at D. Draw PD. PD is the required tangent.



Check: (The check is left to the pupil.)

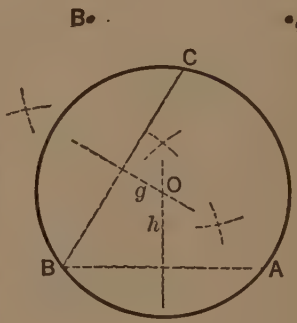
Proposition 13

408. Problem. *To find a circle through three points not in a straight line.*

Given: Three points A, B, and C not in a straight line.

Required: A circle through A, B, and C.

HINT: (See § 217.)



Construction: Draw AB and BC. Find h and g the perpendicular bisectors of AB and BC. Let h and g intersect at O. With O as a center and a radius equal to OA, OB, or OC draw a circle; this circle is the required circle.

Check: (The check is left to the pupil.)

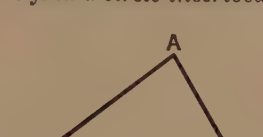
409. Corollary 1 (C N). *To find a circle circumscribed about a given triangle.*

410. Corollary 2. *To find the center of a given circle.*

411. Corollary 3. *To find the center of a given arc.*

Proposition 14 (C N)

412. Problem. *To find a circle inscribed in a given triangle.*

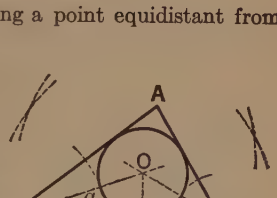


Given: Triangle ABC.

Required: A circle inscribed in $\triangle ABC$.

HINT: What is the proposition concerning a point equidistant from the sides of a triangle? (See § 177.)

Construction: Find g and h the bisectors of $\angle C$ and $\angle B$. Let g and h intersect at O. Find $OP \perp BC$. With O as a center and radius OP draw a circle. The circle will be inscribed in the triangle.



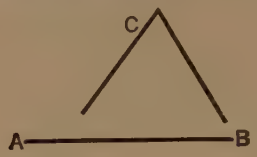
Check: (The check is left to the pupil.)

Construction Exercises. Circles

1. To find an arc of $67\frac{1}{2}$ degrees.
2. To find a circle circumscribed about a given square.
3. To find a circle inscribed in a given square.
4. To find a line which is tangent to a given circle and parallel to a given line.
5. To find a line which is tangent to a given circle and perpendicular to a given line.
6. To find a chord of a circle when the perpendicular from the center to the chord is given.
7. To find a chord of a circle when the position of its mid-point is known.
8. To find a circle tangent to a given circle and having a given center.

Proposition 15

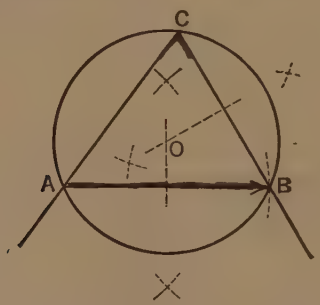
413. Problem. *To find a segment of a circle on a given line and in which a given angle is inscribed.*



Given: Line AB and $\angle C$.

Required: A segment of a circle with AB as its chord and in which $\angle C$ is inscribed.

HINT: Suppose A is taken as a point on a side of $\angle C$ and AB is drawn to the other side. Will the points A, B, and C determine a circle? Will any angle inscribed in the segment ACB equal $\angle C$?



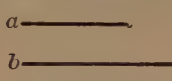
Construction: Take A on a side of $\angle C$ so that CA is less than the given line. With A as a center and a radius equal to AB describe an arc cutting the opposite side at B. Draw a circle through A, B, and C. Segment ACB is the required segment.

Check: (The check is left to the pupil.)

Question: What is segment ACB if $\angle C$ is a right angle? if $\angle C$ is an acute angle? if $\angle C$ is an obtuse angle?

Proposition 16 (C N)

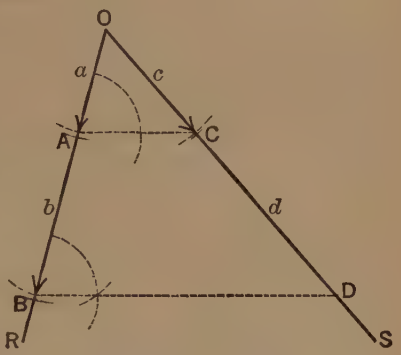
414. Problem. *To find a fourth proportional to three lines.*



Given: Lines a, b, and c.

Required: A line d such that $a : b = c : d$.

HINT: What are the theorems in which four lines form a proportion? Which suggests the simplest construction when a, b, and c are known?

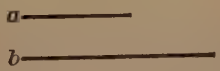


Construction: Draw any angle ROS. Draw $OA = a$, $AB = b$, and $OC = c$. Draw AC. Draw $BD \parallel AC$. CD or d is the required line.

Check: (The check is left to the pupil.)

Proposition 17 (C N)

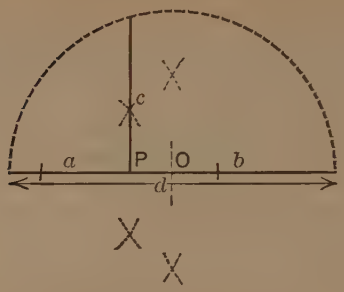
415. Problem. *To find a mean proportional between two lines.*



Given: Lines a and b.

Required: A line c such that $a : c = c : b$.

HINT: Recall the theorems concerning mean proportionals. Which suggests the simplest construction when a and b are known?



Construction: Draw d equal to $a + b$. Bisect line d. With O the mid-point of d as a center describe a semicircle with d as a diameter. Draw $c \perp d$ at P. Then $a : c = c : b$.

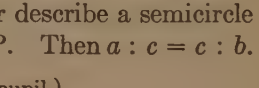
Check: (The check is left to the pupil.)

Proposition 18 (C N)

416. Problem. *To find the segments of a line which are proportional to any number of given lines.*

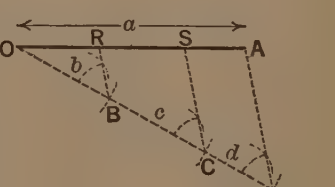
Given: Lines a, b, c, and d.

Required: To divide line a into parts proportional to b, c, and d.



HINT: What is the theorem relating to line segments proportional to any number of other lines? The theorem suggests the following construction.

Construction: Draw $OA = a$. Draw OH making any convenient angle with OA. On OH draw successively $OB = b$, $BC = c$, and $CD = d$. Draw DA. Draw $CS \parallel DA$ and $BR \parallel DA$. Then $\frac{OR}{b} = \frac{RS}{c} = \frac{SA}{d}$.



Check: (The check is left to the pupil.)

417. Corollary (C). *To find any number of equal parts of a line.*

HINT: What is the relation of b, c, and d in this case?

Construction Exercises. Proportional Lines

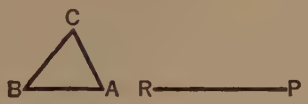
1. To find x if $x = \frac{bc}{a}$ where a, b, and c, are given lines.
2. To find x if $x = \frac{bc}{2a}$.
3. To find x if $x = \frac{b^2}{a}$.
4. To find x if $x = \frac{2b^2}{a}$.
5. To find x if $x = \sqrt{ab}$.
6. To find x if $x = a\sqrt{3}$.

HINT FOR EX. 6: Construct an equilateral triangle whose side is $2a$ and draw the altitude.

7. To find x if $x^2 = cd$.
8. To find x if $x^2 = 2a^2$.

Proposition 19 (N)

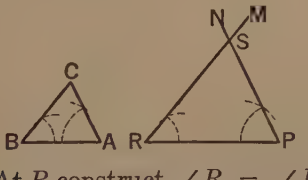
418. Problem. *To find a triangle similar to a given triangle and with a given ratio of similitude.*



Given: $\triangle ABC$ and line PR.

Required: $\triangle PRS \sim \triangle ABC$, with PR and AB corresponding sides.

HINT: What theorem is used most frequently in proving triangles similar?

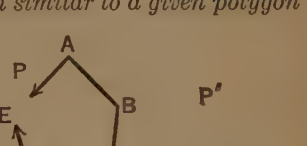


Construction: At R construct $\angle R = \angle B$ and at P construct $\angle P = \angle A$. Let RM and PN intersect at S. $\triangle PRS \sim \triangle ABC$.

Check: (The check is left to the pupil.)

Proposition 20 (C N)

419. Problem. *To find a polygon similar to a given polygon and with a given ratio of similitude.*

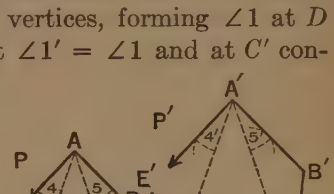


Given: Polygon P and line $C'D'$.

Required: Polygon $P' \sim$ polygon P with $C'D'$ and CD as corresponding sides.

HINT: Divide P into triangles by drawing diagonals and apply § 418.

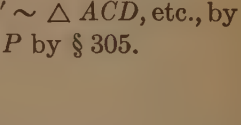
Construction: From a vertex *not at the end* of DC draw diagonals to each of the other vertices, forming $\angle 1$ at D and $\angle 2$ at C. At D' construct $\angle 1' = \angle 1$ and at C' construct $\angle 2' = \angle 2$ and let the sides forming these angles with $D'C'$ intersect at A' thus forming $\triangle A'C'D'$. At A' construct $\angle 5' = \angle 5$ and at C' construct $\angle 6' = \angle 6$, and let the sides forming these angles with $A'C'$ intersect at B' thus forming $\triangle A'B'C'$. In like manner construct a triangle for each of the like-placed triangles in polygon P. Then P' is the required polygon.



Check: $\triangle A'B'C' \sim \triangle ABC$, $\triangle A'C'D' \sim \triangle ACD$, etc., by *a.a.* = *a.a.* Polygon $P' \sim$ polygon P by § 305.

Proposition 21

420. Problem. *To find a square equal to a given rectangle.*



Given: Rectangle R with sides a and b.

Required: A square S equal to R.

HINT: Let x be a side of the square. Then $ab = R = S = x^2$. $a : x = x : b$; that is, x is a mean proportional between a and b.

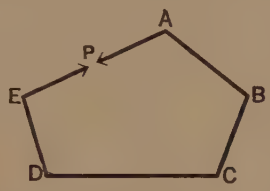
(The construction and the check are left to the pupil.)

421. Corollary. *To find a square equal to a given triangle.*

HINT: $x^2 = \frac{1}{2}ab$. $\frac{1}{2}a : x = x : b$.

Proposition 22

422. Problem. *To find a polygon equal to a given polygon and having one less side.*

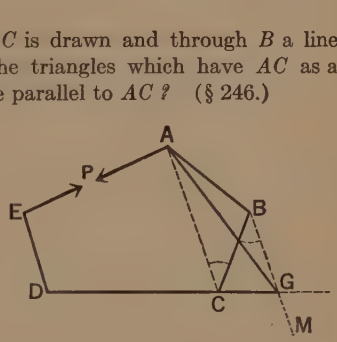


Given: Polygon P with n sides.

Required: A polygon P' = polygon P and having $(n - 1)$ sides.

HINT: In polygon P, if diagonal AC is drawn and through B a line parallel to AC, what can be said of the triangles which have AC as a base and whose vertices lie in the line parallel to AC? (§ 246.)

Construction: In polygon P draw AC. Through B draw $BM \parallel AC$. Produce DC to intersect BM at G. Draw AG. $AGDE\,..\,A$ is the required polygon equal to $ABCDE\,..\,A$ and having one less side.



Check: $\triangle ACG = \triangle ACB$ by § 246. $ACDE\,..\,A = ACDE\,..\,A$. Why? $AGDE\,..\,A = ABCDE\,..\,A$. Why?

423. Corollary 1 (N). *To find a triangle equal to a given polygon.*

HINT: Repeat the process of § 422 until a triangle is obtained.

424. Corollary 2. *To find a square equal to a given polygon.*

HINT: Use §§ 423 and 421.

Construction Exercises. Equal Figures

1. To find a rectangle equal to a given rectangle and having a given base.

2. To find a rectangle equal to a given rectangle and having an altitude one-half as large.

3. To find a triangle equal to a given square and having a given altitude.

4. To find a square equal to the sum of two given squares.
HINT: $c^2 = a^2 + b^2$, by the Pythagorean Theorem.

5. To find a square equal to the difference of two given squares.
HINT: $a^2 = c^2 - b^2$, by the Pythagorean Theorem.

6. To find a square equal to twice a given square.

7. To find a square equal to a given trapezoid.

8. To find an isosceles triangle equal to a given triangle and having an equal base.

9. To find an equilateral triangle equal to the sum of two given equilateral triangles.

Proposition 23 (C N)

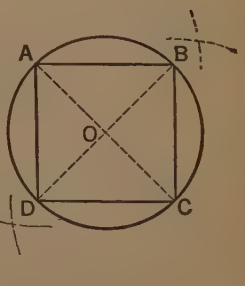
425. Problem. *To find a square inscribed in a given circle.*

Given: Circle O.

Required: A square inscribed in O.

HINT: Are the diagonals of a square equal? How do they meet each other? May diameters be drawn perpendicular to each other?

Construction: Draw diameter $BOD \perp$ diameter AOC. Connect A, B, C, and D. $ABCD$ is the required inscribed square.



Check: (The check is left to the pupil.)

426. Corollary. *To find regular inscribed polygons of 8, 16, 32, 64, etc., sides.*

Proposition 24 (C N)

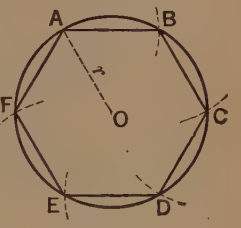
427. Problem. *To find a regular hexagon inscribed in a given circle.*

Given: A circle O.

Required: A regular hexagon inscribed in O.

HINT: How is a side of a regular hexagon related to its radius?

Construction: Draw any radius and call it OA. With A as a center and with a radius equal to OA, or r, draw an arc intersecting the circle at B. Draw chord AB. With B as a center draw chord BC equal to AB. Draw the other chords in like manner and the completed figure is the required inscribed regular hexagon.



Check: (The check is left to the pupil.)

428. Corollary. *To find regular inscribed polygons of 3, 12, 24, 48, etc., sides.*

Proposition 25

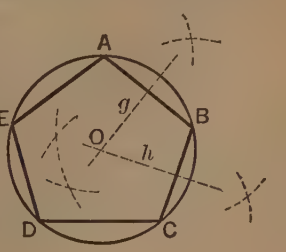
429. Problem. *To find a circle circumscribed about a given regular polygon.*

Given: A regular polygon $ABCDE$.

Required: A circle circumscribed about $ABCDE$.

HINT: The center of the circle must be equidistant from A, B, C, D, E. How can a point be found equidistant from A, B, and C?

Construction: Find g and h, the perpendicular bisectors of two adjacent sides of the polygon. Let g and h intersect at O. The circle drawn with center O and radius OA will be the required circle.



Check: (The check is left to the pupil.)

430. Corollary. *To find a circle inscribed in a regular polygon.*

HINT: If circles are circumscribed about and inscribed in a regular polygon, what is the relation of their centers? (§ 340.) Apply § 429 to find the center. How is the radius of the required circle found?

(This problem is to be solved by the pupil.)

Construction Exercises. Circles and Regular Polygons

1. To find an equilateral triangle inscribed in a circle.

2. To find an equilateral triangle circumscribed about a circle.

3. To find a regular hexagon having a given line as a side.

4. To find a circle whose circumference is three times that of a given circle.

5. To find a circle whose circumference is equal to the sum of the circumferences of two given circles.

6. To find a circle whose circumference shall equal the circumference of a given circle times $\sqrt{2}$.

7. To find a circle equal to four times a given circle. (Area.)

8. To find a circle which shall equal three times the area of a given circle. (Area.)

9. To find a circle equal to the sum of two given circles. (Area.)

10. To divide a given circle into arcs which have the ratio 1 : 2.

11. To find an equilateral triangle by means of a circle and its regular inscribed hexagon.

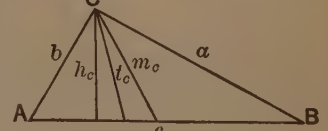
12. To find a square circumscribed about a given circle.

13. To find a regular hexagon circumscribed about a given circle.

14. To find a regular octagon circumscribed about a given circle.

Supplementary Construction Exercises. Triangles

For the purpose of more easily defining construction problems a special notation is frequently used. In $\triangle ABC$ the sides opposite $\angle A$, $\angle B$, and $\angle C$ are called a, b, and c. Any secondary parts drawn to the sides are given corresponding subscript letters. For example, in the annexed figure



h_c, t_c, and m_c are the altitude, the angle bisector, and the median drawn to the side c.

HINT: Before solving the following exercises the pupil should review §§ 403 and 404.

To find $\triangle ABC$ when given:

1. b, c, and m_c.
2. b, c, and h_c.
3. c, h_c, and m_c.
4. b, h_c, and m_c.
5. b, $\angle C$, and t_c.
6. $\angle A$, $\angle C$, and t_c.
7. $\angle A$, $\angle B$, and t_c.
8. h_c, t_c, and $\angle C$.
9. $\angle A$, $\angle B$, and h_c.
10. c, m_a, and m_b.

LOCI

431. Locus of Points. The figure which contains *all the points* which satisfy a given condition (or conditions) and *no other points* is called the *locus of points* which satisfy that condition. The word *locus* comes from a Latin word meaning "place." The plural of locus is *loci* (pronounced lō′sī). The locus of points is therefore *the place of points:* it may be one or more points, lines, or surfaces. Frequently in mathematics the expression "locus of a point" is used in describing the path of a point which moves according to a given fixed condition.

Oral Exercises

1. Describe the locus of points one inch distant from two parallel lines which are two inches apart.

2. Describe the locus traced by the ends of the hands of a watch.

3. If a man walks along a straight street and always keeps five feet from the curb, what is his path or locus?

4. A picture molding is to be placed on the walls of a room eighteen inches from the ceiling. Describe its locus.

432. How to Solve a Locus Problem.

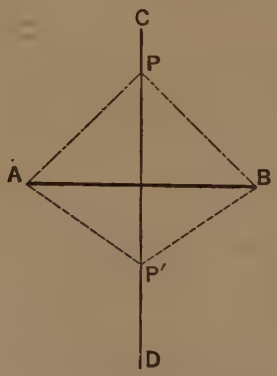
1. Locate a sufficient number of points, which fulfill the given condition, to form an idea of what the locus may be.

2. Prove that every point (that is, *any point*) on the assumed locus satisfies the given condition.
3. Prove that every point (that is, *any point*) which satisfies the given condition lies on the assumed locus.

Parts 2 and 3 are both required in order to satisfy the definition of a locus. Part 2 shows that *all the points* on the assumed locus satisfy the given condition and Part 3 shows that there can be *no other points* which satisfy the condition.

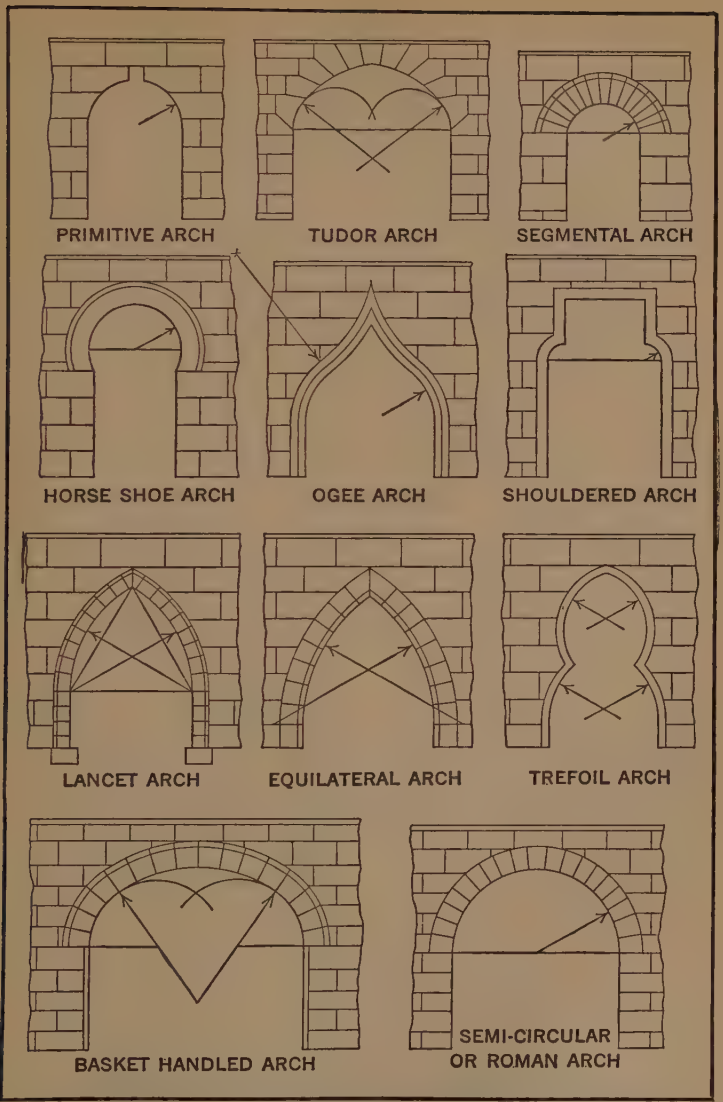
Proposition 1 (C N)

433. Theorem. *The locus of points equidistant from the extremities of a line is the perpendicular bisector of the line.*



Given: CD, the perpendicular bisector of AB.
To prove: CD is the required locus.

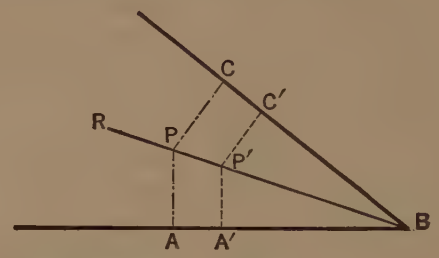
Proof: (A) Let P be any point on CD.
$PA = PB$ by § 168.
(B) Let P' be any point such that $P'A = P'B$.
P' lies on CD by § 170.
Therefore CD is the required locus by definition.



Arches are formed of an arc or of arcs of circles, the lengths of whose radii and the location of whose centers depend upon facts learned in locus problems.

Proposition 2

434. Theorem. *The locus of points equidistant from the sides of an angle is the bisector of the angle.*



Given: BR the bisector of $\angle CBA$.

To prove: BR is the required locus.

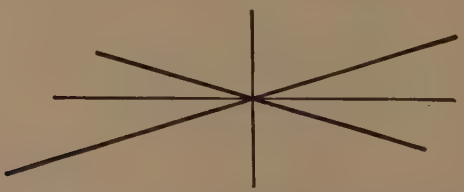
Proof: (A) Let P be any point on BR.
$PA = PC$ by § 172.

(B) Let P' be any point such that $P'A' = P'C'$.
P' lies on BR by § 173.

Therefore BR is the required locus by definition.

435. Corollary (C N). *The locus of points equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by those lines.*

(To be completed by the pupil.)



436. Distances. A locus is frequently determined, wholly or in part, by its distance from some other magnitude; the term *distance* having the meaning of "shortest distance."

1. The distance between two points is the length of the straight line joining the points.

2. The distance from a point to a line is the length of the perpendicular from the point to the line.

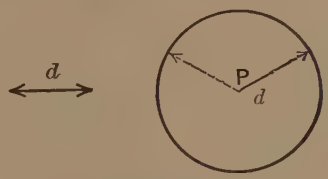
3. The distance between two parallel lines is the length of the segment of a common perpendicular cut off between the two parallels.

4. The distance from a point to a circle is measured on the line which joins the point to the center of the circle and is the length of the segment (internally or externally) between the point and the circle.

5. The distance from a circle to a line is measured on the perpendicular from the center of the circle to the line and is the length of the segment between the circle and the line.

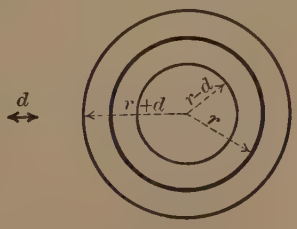
Proposition 3

437. Theorem. *The locus of points at a given distance d from a given point P is the circle having P as its center and d as its radius.* (§ 182 and *Prin.* 32.)



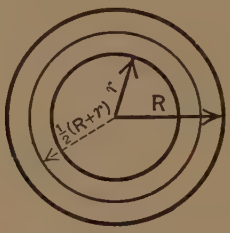
Proposition 4

438. Theorem. *The locus of points at a given distance d from a given circle with radius r is two circles concentric with the given circle and with radii of $r + d$ and $r - d$.*



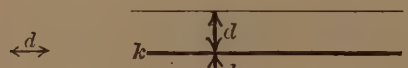
Proposition 5

439. Theorem. *The locus of points equidistant from two concentric circles whose radii are R and r respectively is a concentric circle with a radius of $\frac{1}{2}(R + r)$.*



Proposition 6

440. Theorem. *The locus of points at a given distance d from a given line k is two lines parallel to and on opposite sides of k and d distance from it.*



Proposition 7

441. Theorem. *The locus of points equidistant from two parallel lines is the parallel mid-way between them*

Exercises. Simple Loci

In the exercises which follow, determine the required locus and tell, when possible, upon what theorem (§§ 433–441) it is based.

1. Find the locus of all houses one mile distant from your home.

2. Find the locus of a point on the rim of a flywheel of a stationary engine.

3. Find the locus of the center of a ball as it rolls straight ahead over a level floor.

4. Find the locus of the center of a circle that rolls around another circle. Are there two parts to this locus?

5. Find the locus of the mid-points of equal chords of a circle.

6. Find the locus of the mid-points of the radii of a circle.

7. Find the locus of the center of a circle that passes through two given points.

8. Find the locus of the center of a circle tangent to the sides of an angle.

9. Find the locus of the center of a circle that is tangent to a given line at a given point.

10. If the base of a rectangle has a fixed position and length, but its altitude changes in length, find the locus of the intersection of its diagonals.

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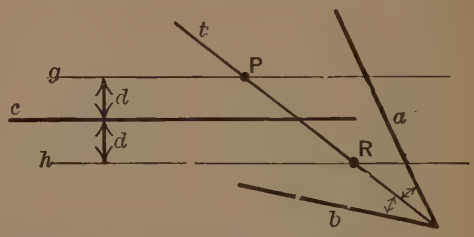
AN ANCIENT ROMAN AQUEDUCT

The Pont du Gard between Nimes and Avignon, France, shows a maximum of strength and a minimum of material due to the use of its many arches. This bridge was erected during the reign of Augustus Cæsar. It is 180 feet high and consists of three tiers of Roman arches. In the complete bridge the lowest tier consists of six arches: one has a span of 75 feet, each of the others measures 60 feet. In the second tier are eleven arches, each with a span of 75 feet. The upper tier, which supported the water-course, consists of thirty-five arches. The aqueducts built by the Romans are their masterpieces of engineering and the Pont du Gard has no rival among the existing remains of Roman times.

11. The hypotenuse of a right triangle has a fixed position and length and its legs change in length. Find the locus of the vertex of the right angle.

442. Compound Loci. The locus of points sometimes satisfies more than *one* condition. Each condition may determine a locus and the point (or points) of the intersection of these loci is the required locus. When a locus is determined by the intersection of two or more loci it is called a *compound locus.*

Illustration: Find all the points which are equidistant from the sides of an angle and at a given distance from a given line.



All points on g and h are at the given distance d from the given line c by § 440.

All points on t, the bisector of the given angle, are equidistant from a and b by § 434.

Therefore the required locus is points P and R.

Discussion: The discussion should tell under what condition there will be two points in the required locus; one point; no points.

Exercises. Compound Loci

In Exercises 1–8, although only one point is required, there may be several which answer the given conditions. These points should be described by the pupil in determining the full locus. The pupil should also note especially the conditions which make the problem impossible.

1. Find a point which is in line b and is c units from point D.

2. Find a point which is in line b and is c units from line d.

3. Find a point which is in line b and equidistant from points C and D.

4. Find a point which is in line b and equidistant from the intersecting lines c and d.

5. Find a point which is in circle C and d units from point G.

6. Find a point which is in circle C and d units from line c.

7. Find a point which is in circle C and equidistant from parallel lines g and h.

8. Find a point equidistant from the sides of an angle and at a given distance from its vertex.

9. Construct a circle which has its center in line b and passes through points C and D.

10. Construct a circle with radius r and which is tangent to two intersecting lines b and c.

11. Construct a circle with radius r, tangent to line b and passing through point C.

12. Construct a triangle if points P, R, and S are the mid-points of its sides.

NOTE: The laying out of arches as illustrated on p. 343 depends upon constructions similar to those of Exercises 9–11.

REVIEW ORAL EXERCISES. DIAGNOSTIC TEST

1. Why is any side of an equilateral triangle greater than an altitude? Why is such a statement not true for any triangle?

2. Why is it shorter to "cut the corner" in crossing a lot than to walk around the two sides?

3. What is the converse of the statement, "If a man lives in Chicago, he lives in Illinois." Is the converse true? What is the opposite statement? Is the opposite statement true?

4. Why is it impossible for the sides of a triangle to be 4, 5, and 10?

5. What two conditions make chords of a circle unequal?

6. If two inscribed angles of a circle have unequal arcs, why are the angles unequal?

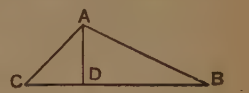
7. If the radii of two circles are unequal, why are their circumferences unequal?

8. Which is the longest line of a right triangle? It is opposite what angle in the triangle?

9. If two corresponding sides of similar triangles are unequal, why are their areas unequal?

10. Combine the expressions $a > b$ and $x = y$ to form five inequalities.

11. Combine the expressions $a > b$ and $c > d$ to form an inequality. Why cannot more than one inequality be formed?



12. If $AD \perp BC$ and $AP > AC$, state the inequalities that follow.

13. What three regular polygons can you construct whose sides equal a given line?

14. Can a given angle be bisected? Can it be trisected?

15. Can a given line be bisected? Can it be trisected?

16. Can you find an angle of 20°? of 15°?

17. Can a triangle be found from any three of its parts? Can you state an exception?

18. Describe what you think is the easiest way of finding a triangle (*a*) similar to a given triangle, (*b*) equal to a given triangle, (*c*) congruent to a given triangle.

19. Describe at least two ways of finding two parallel lines.

20. Describe how to find an arc equal to a given arc.

21. Given the mid-point of a chord of a given circle, explain how to find the chord.

22. If the diameter of a circle is 10 in. describe a method of finding a chord which equals $5\sqrt{3}$ in.

23. If the diameter of a circle is 10, describe a method of finding a chord equal to $5\sqrt{2}$.

24. Describe a method of finding a regular hexagon by the use of the compasses only.

25. What is meant by the statement: "The perpendicular bisector of a line is the locus of points equidistant from the extremities of the line"?

26. What two conditions must be proven when demonstrating the proof of a locus construction?

27. How many points determine a line? a circle? the perpendicular bisector of a line? the bisector of an angle? a median of a triangle?

28. What is a compound locus? Make up a problem of your own the answer of which is found by two intersecting loci.

SUPPLEMENT

THE NEW TYPE OF GEOMETRY EXAMINATION

Much has been done recently to get away from the old standard or formal examination in geometry and to develop a new type of question, which though more informal is believed to be more comprehensive. This new type of examination, besides covering a wide range of the principles and tools of geometry, can be graded more easily and reduces the time of the examination period to a minimum.

Various types of questions are found in these new tests; samples of a few of the more important kinds follow. The tests of Agnes L. Rogers, of Vera Sanford, of Ben D. Wood, of D. G. Paterson, of J. H. Minnick, of Raleigh Schorling, and of others have been suggestive in forming the questions in the different sections. These lists are not exhaustive, of course; they are given merely as an illustration of what is being done. **It is not intended that these groups should be used as actual tests.** They will serve, however, to acquaint the pupil with what might be expected in future examinations and also to bring in new and wider fields for review at the close of a course. Many pupils and teachers will no doubt supplement the questions given here by the addition of those of their own making.

Type A. Statements to be Completed

COPY and fill in the blank spaces in the following sentences so as to make sensible and significant statements. Put one word in each space.

1. The ——— formed ——— two straight lines drawn from the same point is called an ———.

2. When one straight line meets another straight line and makes the adjacent angles ———, these angles are called ——— angles.

3. A bisector divides a magnitude into ——— ——— parts.

4. A sector of a circle is bounded by an ——— and two ———.

5. If two central ——— are equal, their ——— are equal.

6. The ——— of two chords has the ——— distance from the ———.

7. An angle inscribed in a ——— is a right angle.

Type B. General Statements (Composition)

Read the following statements carefully. For each of those which you think is *incorrect* or *insufficient* write a proper statement.

1. A shadow is a surface.

2. A square has four equal sides.

3. A proportion is a fraction.

4. Euclid lived before Archimedes.

5. Vertical angles and adjacent angles are equal.

6. An arc and an arch are the same thing.

7. A circle can be circumscribed about a parallelogram.

8. Descartes was an Englishman.

9. The ratio of the sides of an isosceles triangle is unity.

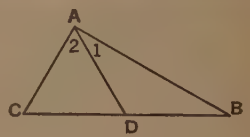
10. A figure two of whose sides are equal is isosceles.

11. The area of a regular polygon equals its perimeter.

12. Angles whose sides are parallel each to each are equal.

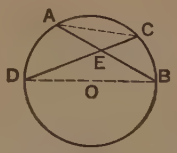
Type C. Conclusions from Figures

Write out the supporting Fact for *each true* conclusion in the following exercises:



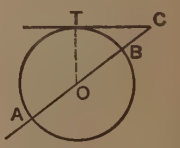
1. In the annexed figure, if $\angle C = 60°$ and $\angle B = 30°$ and $\angle 1 = \angle 2$, then $CD = DB$;

2. $\angle BDA = 105°$.



3. In the annexed figure, $AE \times EC = DE \times EB$;

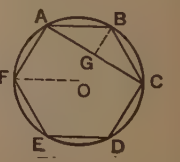
4. $\angle BDA + \angle ACB = 180°$.



5. In the annexed figure, if CT is a tangent and $CBOA$ is a secant to the circle whose center is O, then OTC is a right angle;

6. $CT^2 = CO^2 - OT^2$;

7. $CO : CT = CT : CB$.



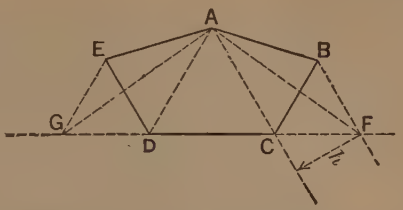
8. In the annexed figure, if $ABCDEF$ is a regular inscribed hexagon, then $BC = OF$;

9. $\angle F = 120°$;

10. $GB = AB\sqrt{2}$.

11. In the annexed figure, if $AC \parallel BF$ and $AD \parallel EG$, then $\triangle BAC = \triangle FAC$;

12. $ABFC = \frac{h\,(BF + AC.)}{2}$



Type D. Converses (True, False, and Prose)

Read each of the following statements carefully. Write out the converse *when true*, neglecting any whose converse is false.

1. A square is a quadrilateral.

2. If two sides of a triangle are equal, the angles opposite are equal.

3. Two equal angles are each equal to 30°.

4. An equilateral quadrilateral is a rhombus.

5. Two equal chords are equally distant from the center.

6. If two triangles are similar, their sides are proportional.

7. An inscribed polygon is regular.

8. If two polygons are similar, they can be decomposed into the same number of like-placed similar triangles.

9. If a, b, and c are the sides of a triangle and $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.

10. A regular polygon is equiangular.

Type E. Definitions (Composition)

For each of the following statements which you think is *incorrect* or *improperly worded*, write out a good definition. Remember that a correct statement is not necessarily a good definition.

1. An equation is a balance between two equal quantities.

2. A chord is like a string to tie things up with.

3. To generate means to make.

4. Bisect means to cut into.

5. A diameter is half a circle.

6. A straight angle is two right angles.

7. Area is the number of square inches a thing has.

8. Formula means a little form.

9. Locus means place.

10. An equilateral triangle is equiangular.

11. A straight line is the shortest distance between two points.

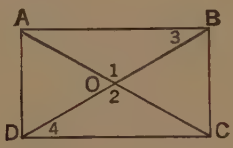
12. A ray is longer than a line segment.

Define the following: secant, ratio of similitude, unity, constants, external segment of a secant, projection, exclusion method of proof, superposition, concurrent lines, nonagon, parenthesis, orthocenter, nonconsecutive, etc. (See Word List, p. 361.)

Type F. Association (Statements and Facts)

In the following exercises for each statement in the left hand column find its corresponding fact from those given in the right hand column:

1. In the rectangle $ABCD$,

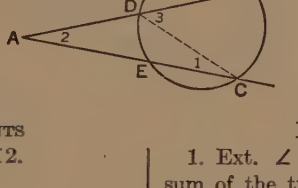


Statements	Facts
1. $AB = DC$.	1. Vertical angles are equal.
2. $\angle 1 = \angle 2$.	2. Opposite sides of a $\square$ are equal and parallel.
3. $\angle 3 = \angle 4$.	3. The diagonals of a rectangle are equal.
4. $AB \parallel DC$.	4. Alt. int. $\measuredangle$ of $\parallel$ lines.
5. $BO = OD$.	5. The diagonals of a $\square$ bisect each other.

2. In $\triangle ABC$, $AB = AC$ and $CD = DB$.

Statements	Facts
1. $AD = AD$.	1. Given.
2. $AC = AB$.	2. Vertical angles are equal.
3. $CD = DB$.	3. *s.s.s.* = *s.s.s.*
4. $\triangle ADC \cong \triangle ADB$.	4. Identical.
5. $\angle C = \angle B$.	5. Base angles of an isos. $\triangle$ are equal.

3. In the annexed figure, AB and AC are secants.



Statements	Facts
1. $\angle 3 = \angle 1 + \angle 2$.	1. Ext. $\angle$ of a $\triangle$ equals the sum of the two opposite int. $\measuredangle$s.
2. $\angle 1 = \angle \frac{1}{2} \overset{\frown}{ED}$.	2. A central angle equals numerically its arc.
3. $\angle 3 = \frac{1}{2} \overset{\frown}{BC}$.	3. Substitution.
4. $\angle 2 = \angle 3 - \angle 1$.	4. An inscribed angle is numerically equal to half its arc.
	5. Subtracting equals from equals.

Type G. Quick Reasoning

From the first ten of the following statements select those which are correct and COPY them. Omit those which are false. In Exercises 11, 12, and 13, COPY those statements which are correct and give the necessary Facts.

1. In plane geometry we deal only with points, straight lines, and plane surfaces.

2. A rectangle is a square if two intersecting sides are equal.

3. Given two isosceles triangles on the same base. The line joining their opposite vertices makes four congruent right triangles.

4. Given four equal chords in a circle. They are all tangent to the same internal circle.

5. $5 : 2 = 4 : 1$ because $5 + 1 = 2 + 4$.

6. If the side of a square is known, it is possible to find the radii of the inscribed and the circumscribed circles.

7. The side of a square equals the diagonal times $\sqrt{2}$.

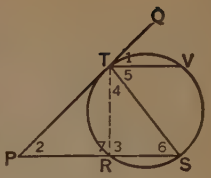
8. The altitude of an equilateral triangle equals its side times $\frac{\sqrt{3}}{2}$.

9. Similar sectors have equal radii.

10. If $BC = \frac{1}{2} AB$ and $AB = XY$, then $\frac{1}{4} XY = \frac{1}{2} BC$.

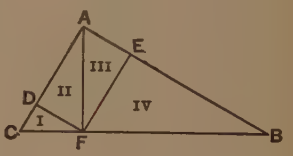
In the following exercises, copy each statement which you find correct and write with each the correct Fact.

11. In the annexed figure, PQ is a tangent at T, PS is a secant intersecting the circle at R and S, $TV \parallel PS$ and TS is a diameter.



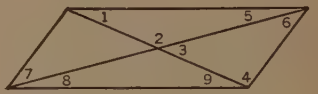
1. $\angle 1 = \angle 5$.
2. $\angle 1 = \angle 2$.
3. $\angle 5 = \angle 6$.
4. $\overset{\frown}{TR} = \overset{\frown}{VS}$.
5. $\angle 3 = \angle 7 = 90°$.
6. $TR^2 = TS^2 + RS^2$.
7. $PT^2 = PR \times PS$.
8. $TR^2 = PR \times RS$.

12. Given rt. $\triangle CAB$ with $AF \perp CB$, $FD \perp CA$ and $FE \perp BA$.



1. $\triangle$ III $\sim \triangle$ I.
2. $\triangle AFB \sim \triangle$ II.
3. $CF : CB = FD : AB$.
4. $FB^2 = EB \times AD$.
5. $DF^2 = AD^2 + DC^2$.
6. There are seven similar triangles.
7. $\triangle DFC \cong \triangle EFA$.

13. Given the parallelogram,



1. $\angle 1 = \angle 4$.
2. $\angle 2 + \angle 3 = \angle 4 + \angle 5 + \angle 6 + \angle 7$.
3. $\angle 1 = \angle 7$.
4. $\angle 4$ is a right angle.
5. $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 + \angle 7 = 180°$.
6. $\angle 7 + \angle 9 + \angle 4 + \angle 8 = 180°$.

Type H. Quick Computation

1. The legs of a right triangle are 6 and 8. Find the hypotenuse.

2. An inscribed angle of 34° has what arc?

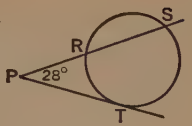
3. Find the area of the sector with a central angle of 60° and radius 5 in. Find the area of the segment with the same parts.

4. In an isosceles triangle one angle is double another. Find the two sets of values of the angles.

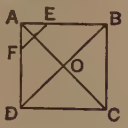
5. Given a regular octagon. Find one exterior angle.

6. One angle of a right triangle is 30° and the side opposite it is 3 in. Find the third angle and the other two sides.

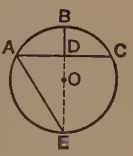
7. Given tangent PT and secant PRS with $\angle P = 28°$ and $\overset{\frown}{TS} = 150°$. Find $\overset{\frown}{TR}$ and $\overset{\frown}{RS}$.



8. Given a square $ABCD$ whose side is 8 in. and $BE = BO$. Find AE. Find FE.



9. In the annexed figure, $BD \perp AC$ and equals 9 in. and $AD = DC = 12$ in. Find the diameter BE and the chord AE of the circle.



10. Find the side of an equilateral triangle whose altitude is 12.

11. Find the area of a parallelogram whose sides are 4 in. and 10 in. and include an angle of 45°; if they include an angle of 30°; if they include an angle of 60°.

12. One leg of a right triangle is 15 and the altitude upon the hypotenuse is 12. Find the other leg and the segments of the hypotenuse.

Type I. Locus Problems

State in a comprehensive sentence or phrase or indicate by a sketch, without proof, the following loci in a plane:

1. The locus of a point equidistant from two given parallel lines.

2. The locus of a point one inch from two given intersecting lines.

3. The locus of a point 2 in. from a given line and 2 in. from a given point.

4. The locus of a point equidistant from two intersecting lines and 2 in. from a given line.

5. The center of the circumscribed circle of a triangle lies on what two intersecting loci?

6. Find the locus of the mid-point of a chord which passes through a given point in a given circle.

7. Show that the right-angled vertex of a right triangle which keeps the same hypotenuse lies in a circle.

8. Discuss the several possible loci of a point 3 in. distant from a given point and 2 in. distant from two given intersecting lines.

9. Find the locus of the vertices of a triangle having a given base and a given altitude.

10. Find the locus of a point R distant from a given circle of diameter d and r distant from a circle of diameter D.

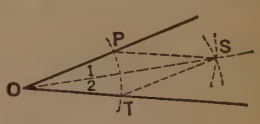
Type J. Association (Terms)

In each line of the following exercises select the term from columns *B*, *C*, and *D* which is most closely associated with the term in column *A*. The selected term does not necessarily define the term in column *A*.

	A	*B*	*C*	*D*
1.	parallelogram	triangle	polygon	rectangle
2.	quadrant	symbol	bisector	arc
3.	diameter	chord	perimeter	tangent
4.	altitude	hypotenuse	perpendicular	vertex
5.	corollary	theorem	conclusion	statement
6.	trapezoid	parallelogram	quadrilateral	isosceles
7.	vertex	reflex	point	degree
8.	decagon	axiom	proposition	polygon
9.	median	bisector	transversal	diagonal
10.	consequent	dimension	denominator	ratio
11.	apothem	perpendicular	segment	problem
12.	rhombus	locus	sector	parallelogram
13.	projection	bisector	arc	line
14.	triangle	arc	circle	diagonal
15.	vertical	oblique	opposite	equal

Type K. Interpreting Constructions

Copy the following constructions carefully—study the relationship of parts—then write out the answers to the following questions:



1. Why does $OP = OT$?

2. Why does $PS = TS$?

3. How is the distance from S to the sides of the angle measured?

4. Why are these two distances equal?

5. Why does $\angle 1 = \angle 2$?

6. $ABCDEF$ is what special figure?

7. Why are $\triangle AEC$ and BOC similar?

8. OHC is what special right triangle?

9. GB is the side of what regular polygon?

10. Why is $\overset{\frown}{BG} = \overset{\frown}{GC}$?

11. Prove $ABCO$ a parallelogram.

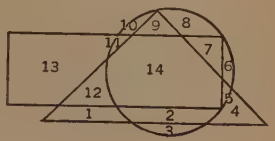
12. Name two pairs of congruent triangles.

13. How many degrees in $\angle OPC$?

14. Why should the line joining P and B meet OC at right angles?

15. Why would the line joining O and D be the perpendicular bisector of EC?

16. In the annexed figure, what numbers are in the rectangle and circle, but not in the triangle?



17. What spaces are in the triangle only?

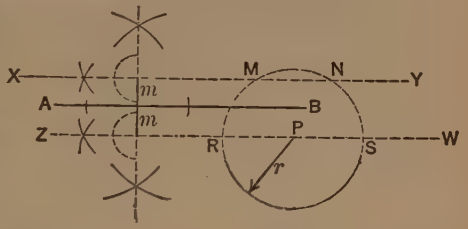
18. What number is in the same geometric figure or figures as 5?

19. What spaces are in the circle only?

20. Describe where the numbers 1, 2, 3, 7, and 11 are found.

21. The lines XY and ZW are the loci of all points at what distance from what line?

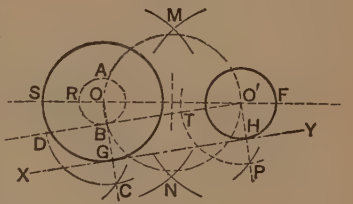
22. The circle $MNSR$ is the locus of what points?



23. The points M, N, S, and R are what locus?

24. Under what conditions would there be only three points in the locus? two points? one point? no points?

25. XY is the common external tangent of circles O and O' with radii OS and $O'F$. How long is RS? OR?



26. Of what purpose are the arcs which intersect at M and N?

27. $\widehat{DC} = \widehat{PT}$ for what purpose?

28. Why is OBO' a right angle?

29. Why is $BGHO'$ a rectangle?

30. Why is XY a common external tangent?

31. Explain how the other common external tangent may be found.

Courtesy Keystone View Co., Inc.

THE CATHEDRAL AND THE LEANING TOWER AT PISA, ITALY

The leaning tower was completed in 1350. Galileo, the great physicist, availed himself of its oblique position in his studies of the laws of falling bodies. The tower is 14 feet out of perpendicular (which is due to the sinking of the foundation) and the extreme height is 183 feet. Can you construct a problem using these measurements?

Type L. Recognition (Multiple Choice)

In each of the following exercises, select *two parts* (from those enclosed in parentheses) which will make the best or the truest statement. Write the completed sentence using "and" to connect the two parts selected.

1. (A square, trapezoid, isosceles triangle, hexagon, equilateral triangle, rhombus) are regular polygons.

2. Equal complementary angles are (adjacent, of the same size, the supplements of each other, 45° angles, vertical angles).

3. Similar triangles (are equal, are mutually equiangular, are congruent, are equilateral, have the same ratio as their perimeters, have their respective sides proportional).

4. In a parallelogram, (the diagonals are equal, the opposite angles are supplementary, the diagonals bisect each other, the opposite sides are equal, the longest side is used as a base).

5. In an isosceles right triangle, (the hypotenuse equals a side times $\sqrt{2}$, the acute angles are adjacent complementary angles, two of the angles are suplementary, the mid-point of the hypotenuse is equidistant from the vertices, each side is half the hypotenuse).

6. If the angles of a triangle have the ratio of 1:2:3, (two of its sides are equal, one angle is a right angle, one side is double another, the angles are supplementary, the sides have the ratio of 1:4:9).

Type M. Recognition (Analogy)

In each of the exercises which follow, the first term *is to* (:) the second term *as* (=) the third term *is to* (:) one of the parts enclosed in the parenthesis. Write the proper statements.

Example: Dog : puppy = cat : (animal, kitten, mammal, quadruped). The fourth term to be used is kitten. Therefore, dog : puppy = cat : kitten.

1. Rectangle : square = parallelogram : (triangle, quadrilateral, rhombus, polygon, figure).

2. Point : line = line : (perimeter, radius, solid, surface, point).

3. Circle : arc = circumference : (diameter, curved line, arc degree, 360°, center).

4. Rectangle : bh = square : (s, s^2, $s\sqrt{2}$, $\sqrt{4\,s}$).

5. 2 : 1 = diameter : (circle, chord, radius, arc, central angle, diagonal).

6. The radius of the circle circumscribed about a regular polygon : the radius of the polygon = the apothem : (the diameter, number of sides of the polygon, the radius of the inscribed circle, the perimeter).

7. Circle : polygon = circumference : (D, $2\,\pi R$, diameter, perimeter, straight line).

A WORD LIST FOR PLANE GEOMETRY

The pupil should learn to spell, pronounce, and use intelligently in sentences the following words:

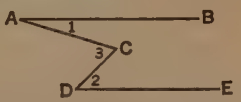
A	symbols	equation	computation	generalize
	solid	surface	plane	dimension
	excess	adjacent	intersect	magnitude
	limited	boundaries	position	
B	extremity	geometric	straight	hypotenuse
	height	lateral	linear	particularly
	coincide	congruent	rectilinear	figure
	segment	quantity	curvilinear	

C	angle	vertex (-ices)	necessary	bisector
	portion	quadrant	sequence	diminish
	center	radius (-ii)	diameter	miscellaneous
	perpendicular	circumference	arc	
D	separate	altitude	significant	arch
	area	respectively	base	auxiliary
	measurement	designs	circle	exclusion
	compasses	radicand	indirect	
E	diagram	scale	measure	ray
	obtuse	reflex	oblique	generate
	coefficient	exponent	subtrahend	multiplicand
	acute	succession	unity	
F	rotation	revolving	exterior	degree
	complements	supplementary	conjugates	similar
	postulate	distance	axiom	construction
	perigon	prolongation	corollary	
G	corresponding	proposition	theorem	assumptions
	pi	authority	argument	demonstrate
	positive	negative	alternation	remainder
	quotient	vertical	difference	
H	square	altering	closed	hypothesis
	analytic	superposition	converse	opposite
	transversal	alternate	rational	isosceles
	synthetic	parallel	scalene	
I	irrational	calculate	equidistant	preceding
	equilateral	equiangular	definition	identity
	consecutive	quadrilateral	trapezoid	parenthesis (-es)
	equidistant	interior	inverse	
J	parallelogram	rectangle	rhombus	nonadjacent
	intersection	mid-point	produce	leg
	diagonal	intercept	nonparallel	polygon
	nonconsecutive	cosine	perimeter	
K	pentagon	hexagon	heptagon	octagon
	proof	dodecagon	pentadecagon	convex
	regular	mutually	locus (loci)	nonagon
	decagon	concave	sufficient	

L	satisfy	concurrent	similarly	absurdity
	terminal	transformation	transposition	translation
	median	concentric	eccentric	numerical
	equivalent	trisect	literal	
M	common	multiple	approximate	constant
	limit	continued	approaches	sector
	extreme	ratio	continuity	major
	variable	mean	minor	
N	inequality	trefoil	quatrefoil	inscribe
	describe	escribe	incenter	circumcenter
	centroid	subtend	chord	composite
	circumscribe	orthocenter	analogous	
O	secant	tangent	contact	ambiguous
	determinate	indeterminate	possible	impossible
	correspondence	similitude	function	sine
	excenter	probable	elementary	
P	antecedent	consequent	proportion	proportional
	similar	projection	semicircle	apothem
	axis (-es)	substitution	formula (-as)	rigor
	harmonic	symmetric	problem	
Q	Ahmes	Euclid	Pythagoras	Pappus
	Hero	Ptolemy	Hippocrates	Archimedes
	Descartes	Saccheri	Plato	Brahmagupta
	Gauss	Thales	Eudoxus	

GENERAL MISCELLANEOUS EXERCISES

1. In the annexed figure, if $\angle 1 + \angle 2 = \angle 3$, prove $AB \parallel DE$.



HINT: Draw a line through C parallel to either AB or DE.

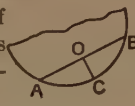
2. On the sides of a square whose side is 5 in., semicircles are drawn externally. Find the total area inclosed.

3. Three points A, B, and C locate the residences of three men who wish to erect a garage which will be equidistant from their homes. Locate the garage.

4. From a point 21 in. from the center of a circle with a radius of 15 in. a secant is drawn. Find the product of the whole secant and its external segment; find the tangent from the point to the circle.

5. In the annexed figure C and G are mid-points of their arcs. Show that $\triangle ODE$ is isosceles.

6. The figure represents a section of a broken wheel or disk. Find the radius if AB measures 16 in. and its perpendicular bisector 4 in.



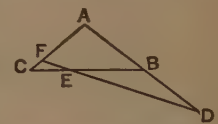
7. An engineer's handbook gives the following formula for the radius of the wheel in the previous figure

$$R = \left[\frac{\left(\frac{s}{2}\right)^2}{h} + h \right] \div 2, \qquad \text{where } s = AB \text{ and } h = OC.$$

Derive the formula.

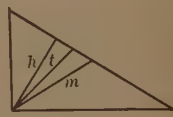
8. If two circles are tangent externally and from any point in the common internal tangent secants are drawn to the two circles, the products of the secants and their external segments are equal.

9. In the annexed figure, if $AB = AC$ and $BE = BD$, prove $\angle AFD = 3\angle D$.



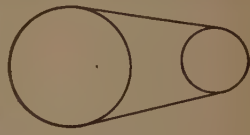
10. If perpendiculars are drawn from the extremities of the base of an isosceles triangle to the opposite sides, the line joining the feet of the perpendiculars is parallel to the base.

11. In the annexed figure, h, t, and m are the altitude, the bisector of the right angle, and the median to the hypotenuse of a right angle. Prove that t bisects the angle included between h and m.



12. A chord 5 in. long cuts a radius two-thirds of its distance from the center at right angles. Find the length of the radius.

13. Two pulleys are connected by an open belt and their diameters are 18 in. and 24 in. If the larger pulley makes 360 revolutions per minute, what is the speed of the smaller pulley?



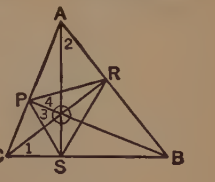
14. Three equal squares are placed side by side to make a rectangle whose perimeter is 80 in. Find the area of a square. Find the diagonal of the rectangle.

15. If each side of an equilateral triangle is trisected and the consecutive points of division joined, a hexagon is formed which is regular and equal to two-thirds the area of the triangle.

16. In $\triangle ABC$ the medians BE and CF intersect at O. Prove that $\triangle BCO$ equals the quadrilateral $AEOF$.

17. The parallel sides of a circumscribed isosceles trapezoid are 6 and 18. Find the radius of the circle.

18. AS, BP, and CR are altitudes of $\triangle ABC$ which intersect at O. PS, RS, and PR join the feet of the altitudes. Prove the following:



a) $\angle 1 = 2$;

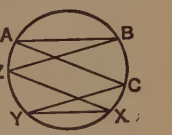
b) a circle with OC as a diameter passes through P and S;

c) $\angle 1 = \angle 3$;

d) $\angle 3 = \angle 4$;

e) Does each altitude of $\triangle ABC$ bisect an angle of $\triangle PRS$?

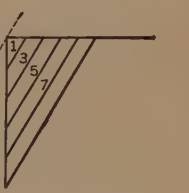
19. The parallel sides of a trapezoid are 20 and 30 and the non-parallel sides are 6 and 8. Produce the non-parallel sides to meet and find the areas of the two triangles and of the trapezoid.



20. If six points A, B, C, X, Y, and Z are located on a circle so that $AB \parallel XY$ and $AC \parallel XZ$, prove that $BZ \parallel CY$.

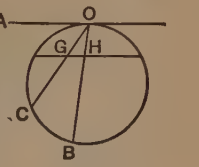
21. If lines are drawn from the extremities of the base of an isosceles triangle perpendicular to the opposite sides, they form an angle supplementary to the angle at the vertex of the triangle.

22. If parallel lines are drawn equally distant from one another as in the annexed figure, prove that the ratio of the successive areas is 1 : 3 : 5 : 7 : etc.



23. A diameter measures 12 in. and is divided into six equal parts by chords perpendicular to it. (*a*) Find the lengths of the five chords. (*b*) Draw chords between the consecutive points of division of the circle and calculate the approximate area of the circle from the triangles and trapezoids formed.

24. In the annexed figure, AO is a tangent parallel to GH. Prove $OC : OB = OH : OG$.



25. Show that the area of a regular hexagon inscribed in a circle is the mean proportional between the areas of the inscribed and the circumscribed equilateral triangles.

26. One base of a trapezoid is 10 in., the altitude is 4 in. and the area is 32 sq. in. Find the length of a line between the legs parallel to the given base and 1 in. from it.

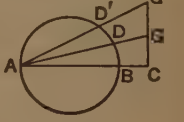
27. Three radii are drawn in a circle of radius $2R$ so as to divide the circle into 3 equal parts. The centers of these radii are taken as centers of arcs of radii equal to R, so as to form a trefoil. Show that the circumferences of the circle and the trefoil are equal. Find the area within the trefoil.

28. If two adjacent sides and one of the diagonals of a parallelogram are 7, 9, and 8 respectively, find the other diagonal.

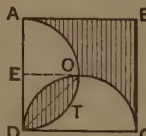
29. A crescent-shaped figure is bounded by a semicircle of radius r and another arc whose center lies on the semicircle produced, as shown in the figure. Find the area and the circumference of the crescent.

30. $ABCD$ is a rectangle having A and C opposite vertices. If P is any point in the plane of the rectangle, show that $(PA)^2 + (PC)^2 = (PB)^2 + (PD)^2$.

31. In the annexed figure, AB is a diameter perpendicular to CG'. Prove $AD \times AG = AD' \times AG'$.



32. Find the area of the shaded surface in the annexed figure. $ABCD$ is a square with side $2R$. AOD and DOC are two semicircles.



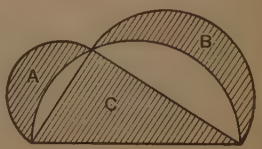
HINT: OTD = quadrant $EOTD - \triangle EOD$.

33. If C is the vertex of the right angle of rt. $\triangle ACB$ and BD is a line cutting AC in D, prove $(BD)^2 + (AC)^2 = (AB)^2 + (DC)^2$.

34. A circular grass plot 12 ft. in diameter is cut by a straight gravel path 3 ft. wide, one edge of which passes through the center of the plot. What is the area of the remaining grass plot?

35. How many degrees in an arc, the length of which is equal to the radius of the circle?

36. Upon the sides of a right triangle C, semicircles are drawn as shown in the annexed figure. Prove $A + B = C$. (Hippocrates' Theorem.)



37. If one leg of a right triangle is the

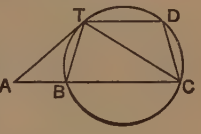
diameter of a circle, the tangent at the point where the circle cuts the hypotenuse bisects the other leg.

38. A hemispherical bowl measuring 12 in. in diameter is filled with water to a depth of 4 in. Find the area of the surface of the water exposed.

39. Given two lines which would meet if sufficiently produced. Construct the bisector of the angle formed between them without producing them to their point of intersection.

40. If from the extremities of a diameter, perpendiculars are drawn to a chord or secant, the feet of the perpendiculars are equidistant from the center.

41. What is the length of the longest rug that can be placed diagonally in a room 12 ft. by 12 ft., if the width of the rug is 5 ft.?



42. If chord TD and secant ABC are parallel and if AT is tangent at T, prove $\triangle ABT \sim \triangle CDT$.

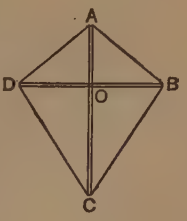
43. Construct a square equal to a given trapezoid. (Show all lines.)

44. Construct a circle equal to one-half a given circle.

APPLICATIONS OF GEOMETRY

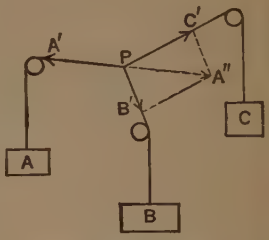
Exercises 1–13 may be used with Chapter One.
Exercises 14–21 may be used with Chapter Two.
Exercises 22–27 may be used with Chapter Three.
Exercises 28–40 may be used with Chapter Four.
Exercises 41–47 may be used with Chapter Five.
Exercises 48–57 may be used with Chapter Six.
Exercises 58–77 are miscellaneous.

1. Robert starts to make a kite by fastening together two light strips of wood AC and BD as a framework. In what special way should he place one of the strips upon the other?



2. The next step in the construction of the kite referred to above is to run a light strong cord connecting the ends A, B, C, and D. What is the relation between the segments of the cord? Why? If the strips are firmly fastened at O, will the cord help to hold them in their proper position? Why?

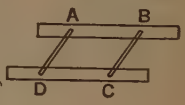
3. If two forces act upon a point at different angles, it is possible to find a force equivalent to the others and the direction in which it acts. In the figure, three different weights $A, B,$ and C, are supported by pulleys and all act upon point P. Either one of the three forces may be considered as exerting a pull equivalent to the combined pull of the other two and could be found (if it were unknown) by the following method: Trace the lines of pull PC' and PB', making them correspond in length to the magnitude of C and B. Complete the parallelogram $PC'A''B'$. Draw the diagonal PA''. Then PA'' corresponds in magnitude to force A, and PA' gives its direction. The above is an application of what is called the *Parallelogram of Forces.*



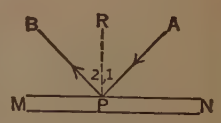
Draw a corresponding figure, if B and C have equal weight and act upon P at an angle of 90°.

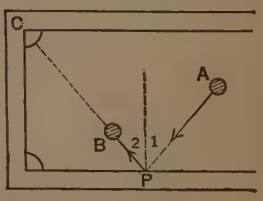
4. John wished to make a square frame. He took four narrow strips of equal length and fastened them together as shown with a nail at each corner. To his surprise the frame would not remain square. Why?

5. A parallel ruler can be made by taking two ordinary rulers and connecting them by strips AD and BC. Describe what special relations must be used in the construction so that the two rulers will always be parallel.

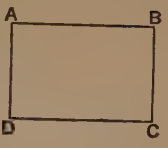


6. It is a law of physics that when a ray of light strikes a reflecting surface it will be reflected so that the angle of incidence equals the angle of reflection. In the figure, the angle of incidence, $\angle 1$, equals the angle of reflection, $\angle 2$ (RP is a perpendicular to the reflecting surface MN). The same law applies to hard bodies rebounding from an elastic surface. The accompanying drawing represents a pool table with balls at A and B. The problem is to make A strike B in such a manner that B will roll into the corner pocket at C. This can be done if A strikes the cushion at P making $\angle 1 = \angle 2$. How may such a point P be found?

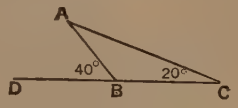




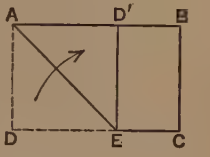
7. $ABCD$ represents the ceiling of a room. An electrician wishes to locate a central spot at which to hang a chandelier. He first ran a cord from A to C and then . . . You explain the rest.



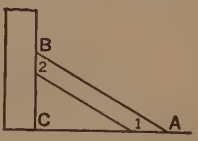
8. A man has a cottage at point B on the shore of a lake. He plans to run a water pipe to an island at A. He observes that $\angle ABD = 40°$. After walking 100 yd. along the shore to C, he observes that $\angle ACD = 20°$. Find the distance to the island from his cottage.



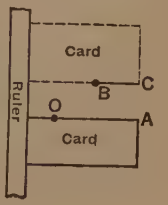
9. A woman has a rectangular strip of cloth $ABCD$ from which she wishes to cut the largest possible square. She folds over a corner so that the edge AD lies along the edge AB and then cuts along the line $D'E$. Why is $ADED'$ a square?



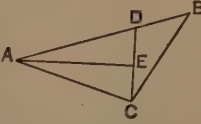
10. A carpenter wishes to brace a partition by fastening a six-foot board (AB) at B, making BC three feet high. At what angle shall he cut the board at A? Why? At what angle shall he cut the other end at B? Why?



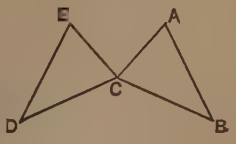
11. Mary was drawing a careful design and found she needed a line through B parallel to OA. She placed a card along OA, then a ruler upright along the card and then moved the card along the ruler until an edge passed through point B. $BC \parallel OA$. Why?



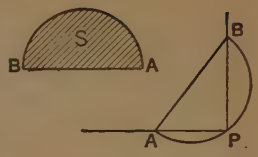
12. A carpenter had an irregular triangular board from which he wished to cut two equal triangular corner braces. He measured AD equal to AC and found that CD was equal to 12 in. At the 6-inch mark he placed point E. Show that $\triangle AED \cong \triangle AEC$. Can this method be used to bisect an angle?



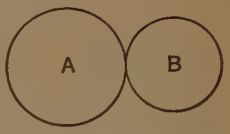
13. A carpenter wishes to know whether the triangular sections ABC and EDC are alike. What measurements should he make?



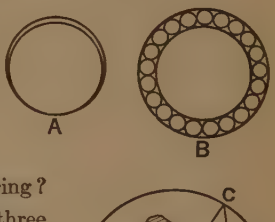
14. George used the following method in drawing a perpendicular to a line at P. He took a semicircular piece of cardboard (S) and placed point A upon the given line. He next moved the semicircle about until it passed through P. Then he marked the position of B and drew BP. BP and AP are perpendicular. Why?



15. If wheel A is geared to wheel B and A is the driver while B is the follower, what advantage is gained by having B smaller than A? If A has 30 teeth and B has 18 teeth, how many revolutions does B make to one of A? If A makes 180 revolutions per minute (R. P. M.), how many R. P. M. does B make? How would the result be changed if a third gear C of 15 teeth were placed between A and B? What is the R. P. M. of gear C?

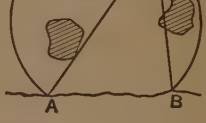


16. In the previous exercise if B is the driver and A the follower, what advantage is gained by having A larger than B? If the number of teeth is 18 and 30, what fractional part of a revolution does A make to one of B? If B makes 120 R. P. M., what is the speed of A?

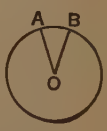


17. Fig. A shows a shaft turning in a plain bearing. Fig. B shows a shaft turning in a roller bearing. What is the advantage of the roller bearing over the plain bearing?

18. At a certain place along the coast three rocky islands are a menace to navigation at night. The government has erected lighthouses at A and B, and every pilot is instructed to keep his vessel far enough from shore to make the angle ACB less than 40°. Explain the geometric application.

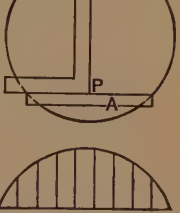


19. A tinsmith had a circular sheet of tin out of which to cut a 36° angle at the center. Not having a protractor he measured the circumference with a cord and found it to be 25 in. Then he measured $2\frac{1}{2}$ in. on the cord, marked it along the edge of the disk (as $\overset{\frown}{AB}$) and cut out the angle AOB. Why did he use $2\frac{1}{2}$ in? Why does $\angle AOB = 36°$?

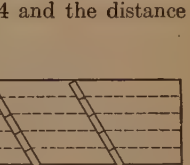


20. If a body is rotated rapidly in a circle and is suddenly released, it tends to fly off at a tangent to the path of the circular motion. This is the principle on which the sling (S) operates.

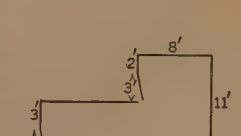
21. A mechanic had to mark the diameter of a circular brass plate. Not knowing the center, he placed a foot-rule (A) across the circle making it measure a 12-inch chord. At the 6-inch mark he placed a square (B) as shown in the figure. Why is PB a part of the diameter? The mechanic could have found the center in at least two ways. Explain how.



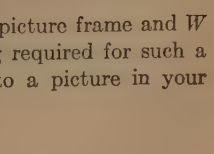
22. The area of a figure may be found approximately by dividing it into sections by means of parallel lines equally spaced. By drawing the chords of the respective arcs, the first and last figures will be triangles, the other figures will be trapezoids. Find the area of the complete figure if the parallel lines are 4, 6, 8, 9, 9, 8, 6, and 4 and the distance between them is 2.



23. When Jane was making her dress she had a piece of cloth to cut into five strips of the same width. Not being able to find a yardstick or measuring device, she took a piece of tape and marked upon it five equal spaces. She placed the tape across the piece of cloth at two different places (as illustrated) and thus obtained five equal widths. What theorem of geometry did she apply?

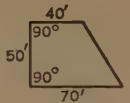


24. Find the area of the room whose floor plan is shown in the figure. How many feet of baseboard are required for this room? Compare the perimeter of the room shown with the perimeter of a rectangular room 11′ x 18′. Compare the area of the room shown in the figure with the area of a rectangular room 11′ x 18′.

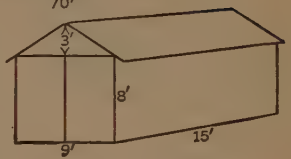


25. Given a and b, the inside dimensions of a picture frame and W the width of the frame. Show that the molding required for such a frame is $2(a + b + 4W)$. Apply this formula to a picture in your home or schoolroom.

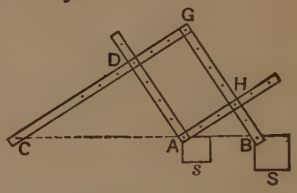
26. A quart of grass seed is sufficient for 300 sq. ft. of lawn. How many quarts would you buy for the plot shown in the figure?



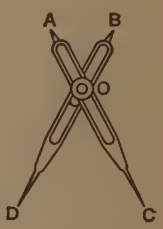
27. A gallon of paint will cover 200 sq. ft. with two coats. How much would you buy to paint the sides and ends of the garage sketched?



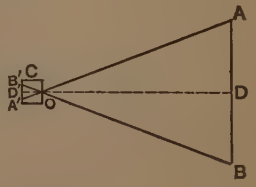
28. The figure illustrates an instrument called a pantagraph which is used for enlarging or reducing a drawing. The instrument turns about a point at C. If point A traces the outline of the small square (s), a pencil inserted at B will trace the larger square (S). For reducing the size of a drawing a pencil at A sketches the form traced by a blunt point at B. Pins at D, G, and H permit the sides of $DGHA$ to vary in length, but $DGHA$ must remain a parallelogram. If, for example, lines are to be enlarged to three times their size, GB is made equal to 3 GH, and CG is made equal to 3 DG while $DG = AH$ and $DA = GH$. If CAB is drawn, what two pairs of similar triangles are formed? Can you explain why the above description is correct?



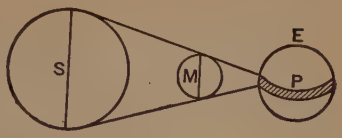
29. The figure illustrates a pair of proportional compasses. Two equal bars are joined by an adjustable thumbscrew at O making $OA = OB$. Then $\triangle OAB \sim \triangle OCD$. Why? $\frac{CD}{AB} = \frac{OC}{OA}$. Why? If the compasses are adjusted so that $\frac{OC}{OA} = \frac{5}{3}$, how will any measurement made with the points C and D compare with a measurement made by A and B?



30. In the figure C represents a camera with film $A'B'$ and lens O. The object to be photographed is AB. If AB is 20 ft., OD' is 6 in., and $A'B'$ is 2 in., find OD. Can you find the range of your own camera by a similar calculation? What geometric fact is used in this application?

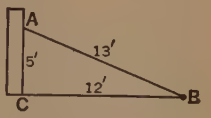


31. S in the figure represents the sun; M, the moon; E, the earth; and P, the path of darkness at the total eclipse of the sun. What distances should be known to find the approximate width of the path? Describe how you might solve this problem.

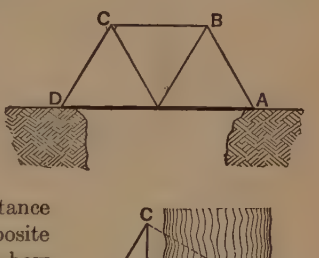


32. Tom and Jack were trying to guess the height of the flagpole on the school campus. Tom observed that the rope for hoisting the flag was hanging by the pole. He walked over to the pole and by two measurements found that the rope was 1 ft. longer than the pole and when stretched out touched the ground 11 ft. from the pole. In three minutes more he had calculated the height of the pole. What was it?

33. A mason used the following plan in laying out a square corner. He tied the ends of a 25-foot line at A and C, 5 ft. apart, and then stretched the line, placing a peg at B, 12 ft. from C. Show that $\angle ACB = 90°$.

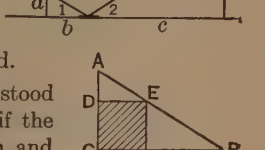


34. The bridge in the figure shows a simple Pratt truss, which is formed of equilateral triangles. AB and CD are called struts, BC is called the topchord. If the height of the bridge is 10 ft., find the lengths of the struts, the topchord, and the bridge.

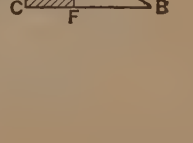


35. An explorer wishes to know the distance between two points D and B on opposite banks of an impassable river. Describe how he might lay out the triangles shown. What measurements should he make in order to calculate DB?

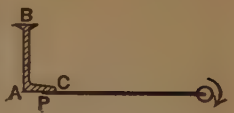
36. An approximate method of measuring the height of an object is to place a mirror horizontally on the ground and stand so that the top of the object is visible in the mirror. Show how the height x can be measured.



37. How large a cubical box $CDEF$ may be stood in the corner closet represented in the figure, if the box occupies the position shown when AC is 21 in. and BC is 28 in.?



38. In the figure, a push rod OP raises valve B by pressing upon its foot AC. If OP is 13 in. and projects 1 in. beyond C, how high may AB be raised if it is held in a vertical position?

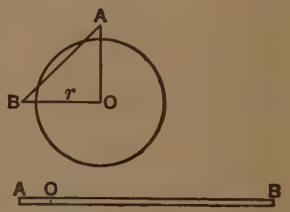


39. A boy plans to attach his aerial to a pole 40 ft. above the ground and run it to his window 20 ft. from the ground. The distance between the house and the pole is 60 ft. How many feet of wire should he buy, allowing 6 ft. for slack and fastening?

40. A coast defense gun has a range of 10 mi. A vessel steams along the shore keeping 6 mi. out and making 20 mi. per hour. How long is the vessel within the range of the gun?

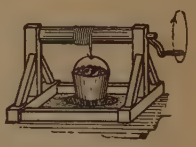
41. A woman has in her yard a circular flower bed 4 ft. across. She decides to double its size. How many feet should be added to the diameter?

42. A sheet metal worker tried to cut out a square equal to a circle by the following method. He drew $OA \perp OB$ making each equal to $\frac{5}{4}\,r$ and used AB as the side of the square. Find his error with a 24-inch circle.



43. A bar as shown in the figure is sometimes used for measuring the motion of a point. The bar turns about O as an axis and $\frac{OB}{OA} = 8$. If B moves $\frac{1}{2}''$ how far does A move? If B turns through 4°, through how many degrees does A turn?

44. It is a law of mechanics that "Power times the distance it moves equals weight times the distance it moves." In the windlass shown in the illustration, the handle moves in a circle whose radius is 10 in., and the diameter of the shaft supporting the bucket is 6 in. If a boy lifts a weight of 60 lbs. in the bucket with what force does he turn the handle?

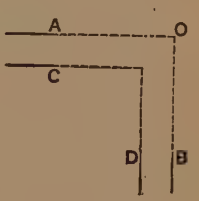


45. Archimedes (250 B.C.) stated that the area of a circle may be obtained by finding the area of a triangle whose base equals the circumference of the circle and whose altitude equals the radius. Show that his method is correct.

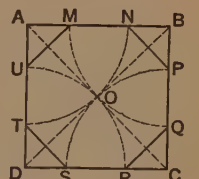
46. The ancient Egyptians found the area of a circle by squaring $\frac{8}{9}$ d. Find the error for an 18-inch circle. To what value of π does the Egyptian method correspond? How is it computed?

47. In the artillery type of automobile wheel there are twelve spokes. Describe how to compute the distance between the centers of the outer ends of any two adjoining spokes. What is the angle included between two such spokes at the center of the wheel? State two ways of locating twelve equally spaced points on a circle.

48. The athletic association of a school decided to construct a cinder running track. One problem involved was how to round a corner. If $OA = OB$, show how to draw an arc of a circle from A to B so as to use the ground space to the best advantage. If $OA = OB = 30$ ft., find the length of arc AB. If the track is 15 ft. wide how much farther does a runner go who follows arc AB than one who follows arc CD?

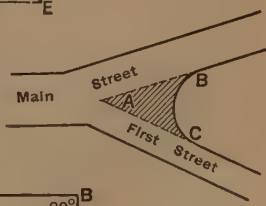


49. The accompanying figure shows how a carpenter cuts a regular octagon from a square board. Explain the process. Is the figure formed a regular octagon?

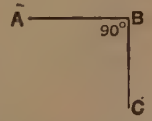


50. Explain how the molding design may be constructed if $BC = CD = 2\,AB = 2\,DE$.

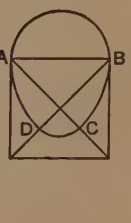
51. To relieve traffic congestion at First and Main Streets, the corner building at A was torn down and the street curb was made to follow an arc of a circle such as BC. How would you draw such an arc?



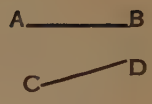
52. A gardener was instructed to set out hedge plants two feet from the lines represented by ABC. Explain how it was done.



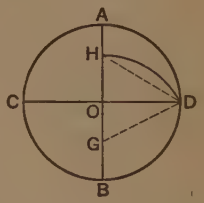
53. Explain the construction of the cross section of the egg-shaped sewer pipe shown in the accompanying figure. ($AB = AC = BD$.)



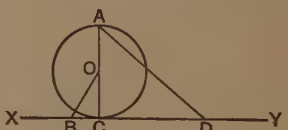
54. How would you locate the centers of all circles tangent to two non-parallel lines, such as AB and CD, without producing those lines so as to intersect?



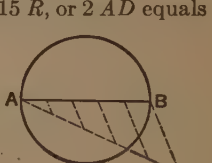
55. The following method is sometimes used to find the side of a regular pentagon and of a regular decagon inscribed in a given circle: In the given circle draw perpendicular diameters AB and CD. With G, the mid-point of OB, as a center and with GD as a radius draw arc HD. Then HD is the side of a regular inscribed pentagon and OH is the side of a regular inscribed decagon. By this means draw a circle and construct a regular pentagram or five-pointed star.



56. The problem of finding a straight line whose length is equal to the circumference of a circle by elementary geometry has been proved to be impossible. The following construction, however, gives a close approximation: Draw $XY \perp AOC$, a diameter of the given circle. Construct $\angle BOC = 30°$. Let $BD = 3R$ and draw AD. $AD = R\sqrt{2\left(\frac{20}{3}\right) - \sqrt{3}} = 3.1415\,R$, or $2\,AD$ equals the circumference of the circle approximately.

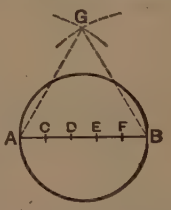


57. A method used in mechanical drawing to inscribe regular polygons of 4, 5, 6, 8, and 10 sides in a circle is illustrated for the regular pentagon as follows:



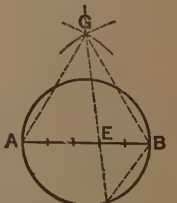
1. Divide the diameter of the circle into as many equal parts as there are sides of the required polygon.

2. Upon the diameter as a side construct an equilateral triangle. ($\triangle ABG$.)



3. Draw a line from the vertex of the triangle to the *second* point of division on the diameter from B and produce it until it cuts the circle. (Line GEH.)

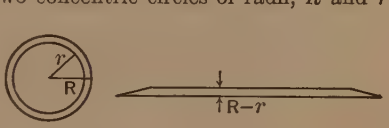
4. Join H and B. Line HB is taken as the side of the required regular polygon inscribed in the given circle.



58. The intersection of the medians of a triangle is called the centroid. Cut out a triangle of cardboard. Find its center of gravity. Why does this triangle balance when a pin is run through the centroid and the triangle is allowed to swing free?

59. The trunk of a tree when trimmed is 48 ft. long. The smaller end is 10 in. in diameter and the larger end 22 in. If the log is cut into 12-foot lengths, what is the largest square stick of timber that can be cut from each?

60. Given a ring formed by two concentric circles of radii, R and r respectively. If the ring can be cut open and spread out as shown in the second figure, show by the formula for the area of a trapezoid that the area of the ring equals $\pi(R + r)(R - r)$.

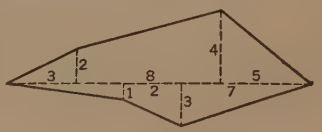


61. The depth of a V-shaped screw thread equals the altitude of an equilateral triangle whose sides equal the pitch of the thread. Show that the depth equals .866 multiplied by the pitch.

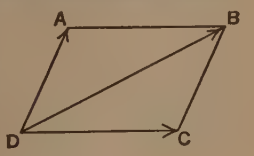
62. Explain how the height of a tree may be found by using an isosceles right triangle.

63. The force P (pressure) of the wind in pounds per square foot is given by the formula $P = .005\ V^2$, where V is the velocity of the wind in miles an hour. What would be the pressure against a billboard 12 ft. high and 50 ft. long when the wind is blowing 30 mi. an hour?

64. The area of a surface with an irregular boundary of straight lines is often found by drawing a line through the surface and using it as a base line to measure off the various offsets to the different vertices. Find the area of the annexed figure.



65. An object at D acted upon by two velocities represented in direction and magnitude by DA and DC will have the resultant velocity represented by the diagonal of the parallelogram DB. Construct the representation of the resultant velocity of two component velocities of 7 and 13, acting upon an object D at an angle of 45° to each other. (See Ex. No. 3.)

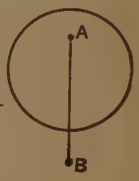


66. How can a sheet-metal worker find the center of a plate in the form of an equilateral triangle? What method may be used if the plate is in the form of a regular hexagon?

67. The historian Thucydides tried to find the area of the island of Sicily from the distance a ship traveled in sailing around it. Was his method correct? With what kind of figures would his method apply?

68. Can a circular glass table-top 7 ft. in diameter be taken through a window opening 3 ft. by 6 ft.?

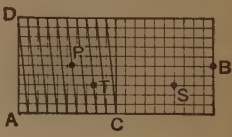
69. A boy finds the center of a large circular piece of cardboard by inserting a pin through any point (A) and allowing the cardboard to turn freely about A. When it comes to rest he draws a line along the plumbline AB. What more must he do to locate the center? Why?



70. If the diameter of the earth is increased one inch, how much is the circumference increased?

71. Two candy boxes each 6 in. by 10 in. have one layer of chocolate wafers of the same thickness. In one box the wafers are 1 in. in diameter, in the other box they are 2 in. in diameter. Which box contains the greater quantity of chocolate?

72. In laying out a tennis court with a steel tape as the only measuring instrument, show how you may check up the square corners. How may you show that the court is rectangular?



73. In the figure above, let $AC = AD = 1$ in. and $PB = 1.45$ in. Find TS.

74. A six-inch water main supplies only one-fourth the required amount of water for a certain town. What is the diameter of a pipe that would supply all the water needed?

75. Allowing 30,000 lbs. to the square inch of cross section for tensile strength, find the load that a round bar one inch in diameter will hold.

76. A mechanic in laying out a 30° angle draws a right triangle whose legs are 4 in. and 7 in. How close does he come to obtaining a 30° angle? Is his method practical for certain cases? (Use Table of Natural Functions.)

77. An earlier settler just before his death said to his sons, "I have buried my money thirty feet from the big chestnut tree and ten feet from the fence that runs by it." Draw the loci to show in how many places the money might have been hidden.

RECREATIONS

(MISCELLANEOUS ARRANGEMENT)

1. An old hermit had a cottage with a window one yard square. It admitted more light than he cared for; so he closed over one-half of it and still had an open square window one yard high and one yard wide. Explain.

2. A gardener has a square plot of ground with 12 posts equally spaced on each side. What is the total number of posts?

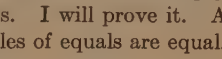
3. Arrange 12 coins on a square with 5 coins on each side.

4. Geraldine had a rectangular piece of paper 16 in. long and 6 in. wide. She cut it into two pieces and by placing them together again formed a rectangular piece 4 in. by 24 in. How did she do it?

5. A camper pitched a square tent between four trees *A*, *B*, *C*, and *D* forming a square. After his departure another camper came along and also pitched a square tent with twice the floor space between the same trees? How was it done?

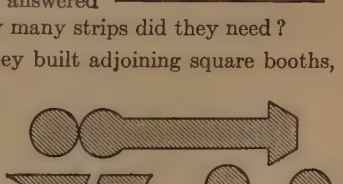
6. Harry and Larry had successfully passed the final examination in geometry and each was served a glass of cold lemonade by Harry's mother. Harry finished his glass while Larry was talking and then said, "Do you know, Larry, that according to geometry my empty glass should be worth as much as your full glass. I will prove it. A glass half empty equals a glass half full. Doubles of equals are equal. Therefore a glass empty equals a glass full."

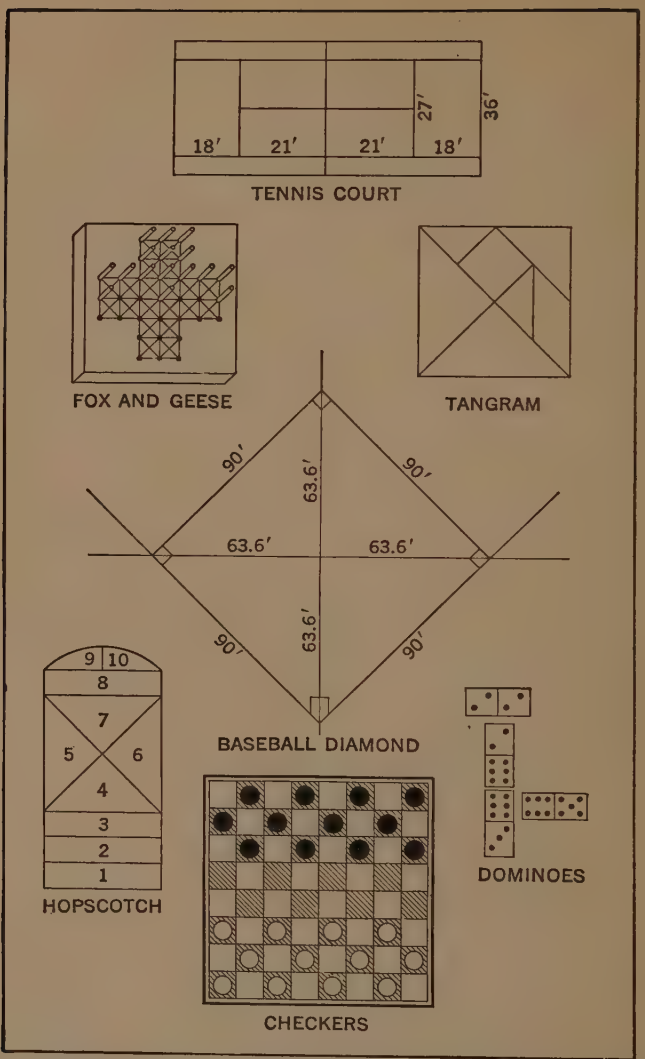
7. Tom and Jerry were going to build the partitions for 12 adjoining square booths of equal size at the church bazaar. Tom said, "I guess that means we must get 48 equal strips." "Oh, no," answered Jerry, "your guess is way off." How many strips did they need?



8. At a previous bazaar where they built adjoining square booths, 49 equal strips were used. What was the number of booths?

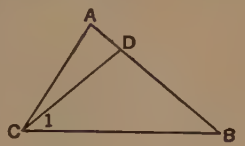
9. The arrow shown here may be made up of the parts which are shown below it. Draw a sketch of the arrow showing on it how the pieces are fitted together. How long did it take you to do it?



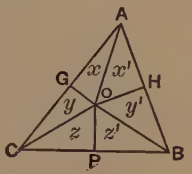


Many games, puzzles, toys, etc., use geometric forms in their pieces, fields of play, and rules. Those illustrated are quite familiar to the average boy or girl, with the exception possibly of the Tangram. The Tangram is a Chinese toy made by cutting a square piece of cardboard into the seven pieces as shown. The object then is to rearrange them to form as many different and interesting designs as possible.

10. Betty and Lettie had finished a course in geometry. "Do you know this one?" asked Betty. "This proves that no triangle can be isosceles. In $\triangle ABC$ draw $\angle 1 = \angle B$. Then $\angle C > \angle 1$ and $\angle 1 = \angle B$ so $\angle C > \angle B$ and $AB > AC$." "I see what's wrong with that," said Lettie, "and I have another one for you." What is wrong with Betty's proof?



11. Lettie's example was an attempt to show that every triangle is isosceles. In any triangle ABC draw OP, the perpendicular bisector of BC. Draw OA bisecting $\angle A$. Draw $OG \perp AC$ and $OH \perp AB$. Draw OC and OB.



rt. $\triangle x \cong$ rt. $\triangle x'$.
rt. $\triangle z \cong$ rt. $\triangle z'$.
rt. $\triangle y \cong$ rt. $\triangle y'$.

$AG = AH$.
$GC = HB$.
$AC = AB$.
$\triangle ABC$ is isosceles.

"That's a good one," said Betty, "I'll give that to my brother; he likes puzzles and such things." "He's only a Freshie. You have to know geometry to get that one," replied Lettie with a superior tilt to her chin. (Lettie is right.)

12. In the magic square shown, the sum of the numbers in each row, each column, and each diagonal is 34. In addition to that, the same total is obtained by finding the sum of the four corner numbers

15	10	3	6
4	5	16	9
14	11	2	7
1	8	13	12

a) in the large square;
b) in each of nine squares, such as 15, 10, 5, 4;
c) in each of six rectangles, such as 15, 10, 8, 1;
d) in each of two rectangles, such as 3, 9, 8, 14;
e) in each of eight trapezoids, such as 10, 3, 7, 14;
f) in each of eight parallelograms, such as 15, 10, 7, 2.

In how many ways is the sum 34 obtained?

Some Eastern peoples wear such magic squares as a talisman or good-luck token.

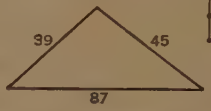
13. The annexed figure shows 13 nails so arranged that if three are removed, only three triangles remain; if four are removed,

two triangles, one trapezoid and one parallelogram remain. How long did it take you to do both?

14. Lillian placed eight pieces on a checkerboard so that no two pieces are in line vertically, horizontally, or diagonally. In other words, she arranged eight queens as in a game of chess so that no queen could take another. How did she do it?

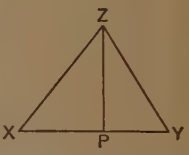
15. Here is the solution of a puzzle. See how well you can word a puzzle of which this is the only solution.

16. By use of Hero's formula (§ 326) find the area of the annexed triangle.



17. In the annexed figures the sections A, B, and C in one case form a square $8 \times 8 = 64$; in the other case by a different arrangement of the same sections they form a rectangle $5 \times 13 = 65$. Apparently $64 = 65$ or by subtraction $0 = 1$.

18. Given a line XY and P any point on the line. With the ends of the line as centers and with any convenient radius draw two arcs intersecting at Z. Draw ZP. $ZX = ZY$; $\angle X = \angle Y$; $ZP = ZP$; $\triangle ZPX \cong \triangle ZPY$; $PX = PY$; therefore any point on the line bisects it.



19. A mechanic was moving a heavy iron beam across the shop floor by means of two 4-inch rollers. The superintendent happened along and asked how far the beam moved to every revolution of the rollers. The mechanic replied that he thought the distance would be the same as the circumference of each roller or $12\frac{4}{7}$ in. The superintendent disagreed. Which was right?

20. When the cake was served for the Sunday-dinner dessert at Mrs. Smith's boarding house, one of the boarders remarked rather sarcastically that he thought Mrs. Smith had cut the cake into the smallest pieces possible. The question then arose: How many pieces could Mrs. Smith have obtained from six straight cuts. Draw a circle and find it out. The cuts, of course, are not all through the center.

Courtesy Keystone View Co., Inc.

THE WILSON DAM

The Wilson Dam in connection with Muscle Shoals in Alabama is one of the most important engineering projects in the United States. Notice the power-cranes used in constructing the dam. In order to lift the boom (or long arm) what side of a triangle shown must be shortened? What angle becomes smaller? What theorem of Inequalities is illustrated? Do the other angles of the triangle change? How? Why?

A BRIEF HISTORICAL SKETCH OF GEOMETRY

The word *geometry* is derived from two Greeks words, *ge* meaning "earth" and *metron* meaning "measure." It is the Greek name for the science which the early Phœnicians and Egyptians had begun and developed about 5000 years ago.

The first important document on mathematics is a manuscript written by an Egyptian priest, called **Ahmes,** about 1700 B.C. This work, called, "Directions for Knowing All Dark Things," is now in the British Museum. It consists of a collection of problems of arithmetic and geometry dealing with the contents of barns, areas of rectilinear figures, and the area of a circular field. In this last problem Ahmes uses a value of π equal to 3.1604. Previous to this time π had been taken equal to 3.

The most important of the early Greek teachers of geometry was **Thales,** who lived from about 640 to 550 B.C. The deductive character of his proofs which he gave to science is his chief claim to distinction. Some of the theorems attributed to him are:

Two angles of an isosceles triangle are equal.

Vertical angles are equal.

A triangle is determined if a side and two adjacent angles are given.

The sides of equilateral triangles are proportional.

A circle is bisected by any diameter.

An angle inscribed in a semicircle is a right angle.

It is stated that on inscribing a right triangle in a circle he celebrated his achievement by sacrificing an ox to the immortal gods.

The most famous pupil of Thales was **Pythagoras,** who was born about 569 B.C. and died about 500 B.C. He founded a school at Crotona, in Italy, which was crowded with enthusiastic students. The students were divided into two classes—beginners and Pythagoreans. The latter formed a secret society which became so powerful that it brought down upon it the hatred of various classes. When the school was investigated by political enemies, Pythagoras and many of his students were murdered by a mob.

Some of the theorems due to Pythagoras are:

The sum of the angles of a triangle equals two right angles.

The plane about a point is completely filled by six equilateral triangles, four squares, or three regular hexagons.

The side and the diagonal of a square have no common measure.

The square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.

This last is called the Pythagorean Proposition. It is one of the most useful truths of all geometry.

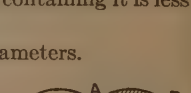
Hippocrates, who was born about 470 B.C., wrote the first elementary textbook on geometry and is credited with first using letters in diagrams to describe a figure. He drew attention to the method of proof called "reductio ad absurdum" or "exclusion," as a proper one for use. His principal discoveries dealt with the theorems on circles among which may be mentioned:

The angle subtended by a chord of a circle is greater than, equal to, or less than a right angle as the segment of the circle containing it is less than, equal to, or greater than a semicircle.

Circles are to each other as the squares of their diameters.

Similar segments are to each other as the squares of their chords.

He also proved that the lune *ADCE* is equal to one-half the isosceles right triangle *ABC* as shown in the annexed figure.



Plato (429–349 B.C.) was primarily a philosopher, like Pythagoras. His interest in geometry was not because of the practical use, but

because of the logic contained in its proofs. An inscription over the entrance to his school read, "Let no one ignorant of geometry enter my door." He systematized the methods of attacking mathematical questions and directed attention to the value of analysis. With him began the careful compilation of definitions, axioms, and postulates. He suggested that geometry should deal only with such figures as could be constructed by an unmarked straightedge and compasses.

Eudoxus (408–355 B.C.) discovered most of what is now called the fifth book of Euclid (i.e., Proportions) and proved his propositions in much the same manner as they are given by Euclid. In this connection he dealt with some theorems dealing with the extreme and mean ratio of a line, sometimes called "the golden section." He wrote the first textbook on solid geometry and discovered some proofs relating to the volumes and surfaces of solids.

Euclid (330–275 B.C.) was a teacher of geometry at Alexandria. It is said that he was gentle and modest of manner. To him is attributed the saying, "There is no royal road to geometry." He appreciated, as did Plato, the cultural value of geometry, as is illustrated by the following story: A boy who had just begun geometry asked, "What do I gain by learning all this stuff?" Euclid replied that knowledge was worth gaining for its own sake, but had a servant give the boy some coppers, "since," he said, "he must make a profit out of what he learns."

Euclid's reputation rests mainly upon his writing, one of the most important textbooks ever written, called *The Elements*. The geometrical part of the book is mainly a compilation from the works of previous writers. The substance of Books 1 and 2 is probably due to Pythagoras; that of Book 3 to Hippocrates; that of Book 5 to Eudoxus; and the greater part of books 4, 6, 10, and 12 to other Greek teachers and writers. *The Elements* of Euclid has been used as a basis for almost all textbooks on geometry since his time.

Archimedes (287–212 B.C.) was one of the most learned men of ancient times. He invented many forms of machines and discovered and wrote about many proofs of mathematics. He lived at Syracuse, in Sicily, and was killed during the sack of that city after its capture by the Romans. It is said that a soldier entered his room while Archimedes was studying a geometrical diagram drawn in sand on the floor. Archimedes ordered him off the diagram that he might not spoil it. The soldier felt insulted and, not knowing who the old man was, killed him.

Archimedes proved that the area of a circle equals $\frac{1}{2}r(2\pi r)$. He

showed the value of π to be less than $3\frac{1}{7}$ and more than $3\frac{10}{71}$. He constructed the table of chords which is essentially a table of natural sines and proved many important propositions dealing with spheres and cylinders in solid geometry.

Hero, who lived during the first century after Christ, cared only for the practical applications of geometry and nothing for its logical accuracy. His principal work consists of finding the areas of fields; propositions on finding the volumes of certain solids; a rule for finding the height of an inaccessible object; tables of weights and measures; and the formula $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$ when s is the semi-perimeter of the triangle and a, b, and c the sides.

Ptolemy, who lived at Alexandria, died A.D. 168. He is the most famous writer on astronomy the the world has known. His great work on this subject is called *The Almagest.* It shows that Ptolemy was a geometrician of the first rank. He wrote a number of short treatises dealing with the fundamentals of geometry. In one he proposed to cancel Euclid's postulate on parallel lines, and in another he wrote to show that there could not be more than three dimensions in space.

Pappus, the last important Greek mathematician, lived about the end of the third century. The most important work of Pappus was a collection of mathematical papers arranged in eight books and we rely on it for our knowledge of the works of others now lost.

He discovered that the volume generated by the revolution of a curve about an axis is equal to the product of the area of the curve and the length of the path described by its center of mass; and that the surface is equal to the product of the perimeter of the curve and the length of the path described by its center of mass. Another of his theorems is a generalized form of the Pythagorean proposition.

Brahmagupta, a Hindu, born 598, proved the Pythagorean theorem, gave a rule for the area of a quadrilateral inscribed in a circle in terms of its sides (which is: Area $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$), and discovered that the area of a circle is equal to that of a rectangle whose sides are the radius and the semi-circumference.

Descartes (1596–1650), the most important contributor to the theory of geometry in modern times, published a small work which he called *La Geometrie*, in 1637. This is the basis for what is now called analytic geometry, a subject which has revolutionized the methods of all mathematics.

Gauss (1772–1855) made many investigations into the theory and subject matter of mathematics. He discovered that it is possible to construct regular polygons of $2^m \times (2^n + 1)$ sides (where m and n are

integers and $(2n + 1)$ is prime), with ruler and compasses. He also spent a great deal of time trying to clarify the assumptions of geometry and to reduce their interdependence.

Non-Euclidean Geometries

A sketch of this kind would not be complete unless some mention were made of the nature and development of non-Euclidean geometry. The earliest conception of this subject was due to the discovery made independently by **Saccheri, Lobatschewsky,** and **J. Bolyai,** that a consistent system of geometry of two dimensions can be produced on the assumption that the axiom on parallels is not true, and through a point any number of straight lines can be drawn parallel to a given straight line. This results in a geometry called *hyperbolic.*

Courtesy Keystone View Co., Inc.

THE BRIDGES OF ST. JOHNS

This picture of the bridges over St. Johns River, Florida, shows three different types of movable bridge structures. The single-track span upon which the train stands is the revolving "turntable" type. It is now replaced by the new double-track bascule bridge span which appears to the right and is shown at an almost vertical position. In the background is the Duval County highway bridge, the center span of which is lifted so that its roadbed always remains in a horizontal position. The construction and operation of these three bridges suggests a great many Facts of geometry. Why do so many triangles appear? What lines must be equal? perpendicular? parallel? What loci are generated? What angles and what lines change their sizes? What properties of right triangles are involved? How many Facts can you find?

Riemann later showed that still another consistent system of geometry of two dimensions can be constructed in which all straight lines are of a finite length. This results in a geometry called *elliptic*. In this geometry no straight line can be drawn which, if produced far enough, will not meet any other straight line. This can be compared with the geometry of figures drawn on the surface of a sphere.

Thus, according as no straight line, or only one straight line, or any number of straight lines can be drawn through a given point parallel to a given straight line, we have three systems of geometry of two dimensions known respectively as elliptic, Euclidean, and hyperbolic.

NOTE: *A Short History of Mathematics* by W. W. R. Ball, *A History of Mathematics* by F. Cajori, and Chapter 3 of *The Teaching of Geometry* by David Eugene Smith have been drawn upon in arranging this brief sketch.

DEFINITIONS

This list of definitions is given for handy reference and for the purpose of aiding in rapid review work. The pupil should always give the definition of a term in a *complete sentence*.

Acute angle	An angle between 0° and 90°.
Acute triangle	A triangle whose angles are acute.
Adjacent angles	Two angles with a common vertex and a common side between them.
Alternate exterior angles	A pair of non-adjacent exterior angles on opposite sides of a transversal.
Alternate interior angles	A pair of non-adjacent interior angles on opposite sides of a transversal.
Altitude of a parallelogram	The perpendicular between its bases.
Altitude of a trapezoid	The perpendicular between its parallel sides.
Altitude of a triangle	The perpendicular between a vertex and its opposite side.
Angle	The figure formed by two rays with common end points.

Antecedent	The numerator of a fraction.
Apothem (regular polygon)	The radius of the circle inscribed in the polygon.
Arc	A portion of a circle.
Area of a surface	The number of square units of measure in the surface.
Axiom	A general statement assumed to be true.
Base of a triangle	Any side of the triangle.
Base of a trapezoid	Either of its parallel sides.
Bisect	To divide into two equal parts.
Broken line	A line composed of successive straight lines not in a straight line.
Center of a circle	The point equidistant from every point of the circle.
Center of a regular polygon	The center of its inscribed or circumscribed circle.
Central angle	An angle formed by two radii.
Chord	A line segment with its ends in a circle.
Circle	A closed curve everywhere equidistant from a point.
Circumference	The linear measure of a circle.
Circumscribed circle	One which passes through each vertex of a polygon.
Circumscribed polygon	One whose sides are tangent to a circle.
Coincide	To correspond exactly.
Complementary angles	Two angles whose sum equals 90°.
Concentric circles	Circles with the same center.
Concurrent lines	Lines which meet in a point.
Congruent figures	Figures which agree in shape and size.
Consequent	The denominator of a fraction.
Constant	A value which does not change.

Converse theorem	One which has significant statements of hypothesis and conclusion interchanged with another.
Corollary	A theorem or problem closely related to another.
Corresponding parts	Parts taken in the same order.
Cosine of an angle	In a right triangle, the ratio of the side adjacent to the angle to the hypotenuse.
Curved line	A line no part of which is straight.
Curvilinear figure	One bounded by a curve or curves.
Decagon	A polygon of ten sides.
Degree (angle)	$\frac{1}{90}$ of a right angle.
Degree (arc)	$\frac{1}{360}$ of a circumference.
Diagonal	A line joining two non-consecutive vertices of a polygon.
Equiangular polygon	One whose angles are equal.
Equilateral polygon	One whose sides are equal.
Equivalent figures	Figures equal in area.
Exterior angle of a polygon	The angle included between a side of a polygon and an adjacent side produced.
Exterior interior angles	A pair of non-adjacent angles, the one exterior and the other interior on the same side of a transversal.
External tangent	The common tangent of two circles which does not pass between them.
Extremes	The first and the last terms of a proportion.
Fourth proportional	The last term of a proportion of four different terms.

Geometry	The subject which deals with points, lines, surfaces, solids, and their relations.
Hexagon	A polygon of six sides.
Hypotenuse	In a right triangle, the side opposite the right angle.
Inscribed angle	An angle formed by two chords intersecting on a circle.
Inscribed circle	A circle tangent to each side of a polygon.
Inscribed polygon	A polygon whose sides are chords of the same circle.
Intercept	To cut off.
Internal tangent	The common tangent of two circles which passes between them.
Isosceles trapezoid	One whose non-parallel sides are equal.
Isosceles triangle	One with two equal sides.
Locus of points	A place of points which answers certain conditions.
Means	The second and third terms of a proportion.
Mean proportion	A proportion with equal means.
Mean proportional	Either mean of a mean proportion.
Median (triangle)	The line joining a vertex to the middle of its opposite side.
Median (trapezoid)	The line joining the mid-points of the two non-parallel sides.
Oblique lines	Lines which form unequal adjacent angles.
Obtuse angle	An angle between 90° and 180°.
Obtuse triangle	A triangle with one obtuse angle.
Octagon	A polygon of eight sides.
Parallel lines	Two lines in the same plane which have no common point if produced indefinitely.

Parallelogram	A quadrilateral whose opposite sides are parallel.
Pentagon	A polygon of five sides.
Perimeter	The linear measure of a polygon.
Perpendicular lines	Two lines which meet so that two equal adjacent angles are formed.
Polygon	A closed broken line.
Postulate	An assumption of the possibility of performing a certain geometric operation.
Problem	A question for solution.
Projection of a line upon another line	The segment of the second line between the feet of the perpendiculars dropped from the ends of the first line.
Proportion	The expression of equality of two equal fractions.
Proposition	The general name for a problem or theorem.
Quadrant	One-fourth of a circle.
Quadrilateral	A polygon of four sides.
Radius (circle)	A line from the center to the circle.
Radius (regular polygon)	The radius of its circumscribed circle.
Ratio	A fraction. The expressed division of one quantity by another.
Ray	A line which begins at a point and extends indefinitely in one direction from it.
Rectangle	A parallelogram with a right angle.
Rectilinear figure	One bounded by straight lines.
Regular polygon	An equilateral and equiangular polygon.
Rhombus	A parallelogram with two intersecting sides equal.

Right angle	An angle with perpendicular sides.
Right triangle	A triangle with one right angle.
Scalene triangle	A triangle with unequal sides.
Secant	A line passing through a circle in two points.
Sector	A figure bounded by two radii and their intercepted arc.
Segment of a circle	A figure bounded by a chord and its arc.
Segment of a line	The part of a line lying between any two of its points.
Semicircle	Half a circle.
Similar arcs	Arcs with the same valued central angles.
Similar polygons	Polygons which are mutually equiangular and whose corresponding sides are proportional.
Similar sectors	Sectors with the same valued central angles.
Similar segments	Segments with the same valued central angles.
Sine of an angle	In a right triangle, the ratio of the side opposite the angle to the hypotenuse.
Square	A rectangle with two intersecting sides equal.
Straight angle	An angle equal to double a right angle.
Supplementary angles	Two angles whose sum equals 180°.
Tangent of an angle	In a right triangle, the ratio of the side opposite the angle to the side adjacent to the angle.
Tangent to a circle	A line which touches a circle in only one place.
Tangent circles	Circles which are tangent to the same line at the same point.

Theorem	A statement to be proved.
Third proportional	The fourth term of a mean proportion.
Trapezoid	A quadrilateral with two sides parallel.
Triangle	A polygon of three sides.
Variable	A value which changes.
Vertex of an angle	The point of intersection of the sides of the angle.
Vertical angles	A pair of non-adjacent angles formed when two lines intersect.

FORMULAS OF PLANE GEOMETRY

Notation

a, b, c sides of a right triangle with c the hypotenuse

$a, b, c, \ldots$ sides of polygon $ABC\ldots$

b base of a figure

b, b' bases of a trapezoid

C circumference

D diameter

d diagonal of a square

h altitude

P perimeter

π 3.1416 or $3\frac{1}{7}$

R radius; radius of circumscribed circle

r apothem; radius of inscribed circle

S area

s $\frac{1}{2}(a+b+c)$

s_n side of a regular polygon of n sides.

Lengths of Lines

Right triangle, $c^2 = a^2 + b^2$.

Equilateral triangle, $h = \frac{s\sqrt{3}}{2}$.

$s_3 = R\sqrt{3}$.

Square, $d = s_4\sqrt{2}$.

$s_4 = R\sqrt{2}$.

Hexagon, $r = \frac{R\sqrt{3}}{2}$.

$s_6 = R$.

Circle, $D = 2R$.

$C = 2\pi R = \pi D$.

$\text{Arc} = \frac{\text{central } \angle}{180} \times \pi R$.

Similar polygons,

$$\frac{P}{P'} = \frac{a}{a'} = \frac{b}{b'} = \text{etc.}$$

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \text{etc.}$$

$$\frac{a}{a'} = \frac{\sqrt{S}}{\sqrt{S'}}$$

Regular polygons of the same number of sides. $\frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}$.

Circles, $\frac{R}{R'} = \frac{D}{D'} = \frac{\sqrt{S}}{\sqrt{S'}}$.

Areas

Triangle, $S = \frac{1}{2} bh.$
$= \sqrt{s(s-a)(s-b)(s-c)}$

Equilateral triangle,
$S = \frac{1}{4} s^2 \times \sqrt{3}.$

Rectangle, $S = bh.$

Square, $S = s^2.$

Parallelogram, $S = bh.$

Trapezoid, $S = \frac{1}{2} h(b + b').$

Regular polygon, $S = \frac{1}{2} rP.$

Circle, $S = \pi R^2 = \frac{1}{4} \pi D^2.$

Sector, $S = \frac{central \angle}{360} \times \pi R^2.$
$= \frac{1}{2} R \times arc.$

Segment,
$S = Sector \mp Triangle.$

Similar polygons,
$\frac{S}{S'} = \frac{a^2}{a'^2} = \frac{b^2}{b'^2} = \text{etc.}$

Regular polygons of the same number of sides. $\frac{S}{S'} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}.$

Circles, $\frac{S}{S'} = \frac{R^2}{R'^2} = \frac{D^2}{D'^2}.$

The following formulas, although not required by the National Committee or by the College Entrance Examination Board, are sometimes demanded for examination purposes by certain schools. Pupils who must meet such requirements should memorize these forumlas and be able to apply them in computations. They are all given with respect to the same side c of $\triangle abc$, and in computations side c may be assumed as a horizontal base line in the figure. The formulas with respect to side a or side b may be written by analogy.

The following additional notation will be needed:

h_c = altitude to side c. t_c = bisector of C.

m_c = median to side c. $p^b{}_c$ = projection of b upon c.

$$h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

$$m_c = \tfrac{1}{2}\sqrt{2(a^2+b^2) - c^2}.$$

$$t_c = \frac{2}{a+b}\sqrt{abs(s-c)}.$$

$$p^b{}_c = \mp\left(\frac{b^2 + c^2 - a^2}{2c}\right).$$ (According as $\angle A$ is *acute* or *obtuse*.)

$$a^2 = b^2 + c^2 \mp 2cp^b{}_c.$$ (According as $\angle A$ is *acute* or *obtuse*.)

PRINCIPLES OF GEOMETRY

The principles which follow are those which are assumed from time to time in this text. Those marked with an asterisk (*) are suggested by the National Committee under the title of "Assumptions and Theorems for Informal Treatment." A few of these principles may be given a formal proof.

Equal Quantities

1. Any magnitude equals itself. (Identity.)

2. Two magnitudes are equal if their ratio is unity, and conversely.

3. Magnitudes equal to the same magnitude (or equal magnitudes) are equal. (Substitution.)

4. A magnitude equals the sum of its parts.

5. If equals are added to equals, the *sums* are equal. (Axiom of Addition.)

6. If equals are subtracted from equals, the *differences* are equal. (Axiom of Subtraction.)

7. If equals are multiplied by equals, the *products* are equal. (Axiom of Multiplication.)

8. If equals are divided by equals, the *quotients* are equal. (The division by 0 is not permitted.) (Axiom of Division.)

NOTE: Special cases of Principles 7 and 8 are, "Doubles of equals are equal" and "Halves of equals are equal."

9. Equal powers or equal roots of equals produce *results* which are equal.

Lines (See also No. 30.)

10. Through one point any number of straight lines can be drawn; and from a point any number of straight lines can be drawn equal to a given line.

*11. A line segment can be produced from either end to any desired length.

12. A line segment can be divided into an integral number of equal parts.

*13. One straight line and only one can be drawn from one point to another. (That is, two points determine a straight line.)

*14. Through a point one line and only one can be drawn parallel to a given line. (Postulate of Parallels.)

*15. Through a point one line and only one can be drawn perpendicular to a given line.

*16. Two lines parallel to the same line are parallel.

*17. Two lines in the same plane perpendicular to the same line are parallel.

18. Two lines intersect, if the interior angles on the same side of a transversal to them are not supplementary.

19. Two lines intersect, if they are perpendicular to two intersecting lines.

20. Two lines can intersect in only one point.

*21. A straight line intersects a circle in at most two points.

*22. A straight line is the shortest line between two points.

*23. The perpendicular from a point to a line is the shortest line between the point and the line.

Angles

24. An angle can be constructed equal to a given angle.

25. An angle can be bisected.

*26. Right angles are equal.

*27. Vertical angles are equal.

*28. Equal angles have equal complements and equal supplements. (Referring to the same angle this is stated, "Complements or supplements of the same angle are equal.")

29. If two adjacent angles have their exterior sides in a straight line, they are supplementary.

*30. If two adjacent angles are supplementary, their exterior sides form a straight line.

31. A central angle equals numerically its intercepted arc.

Circles (See also No. 21 and No. 31.)

*32. One circle and only one can be drawn with any point as a center and with a radius of any given length.

33. Radii of the same circle (or of equal circles) are equal.

34. A point is within, on, or without a circle, according as its distance from the center is less than, equal to, or greater than a radius.

35. Two circles are equal if their radii are equal.

36. A diameter bisects a circle.

Special

*37. Any geometric figure may be moved from one position to another without change either of shape or of size. (Postulate of Mobility.)

38. Corresponding parts of congruent figures are equal.

*39. The area of a rectangle equals a base times its altitude.

Variables

If the number of sides of an inscribed or circumscribed regular polygon be increased *without bound*,

40. the radius of the circle may be substituted in an equation for the apothem of the inscribed polygon;

41. the circumference of the circle may be substituted in an equation for the perimeter of either polygon;

42. the area of the circle may be substituted in an equation for the area of either polygon.

Unequal Quantities (See also No. 22, No. 23, and No. 34.)

43. A magnitude may be substituted for its equal in an inequality.

44. If unequals are operated upon by equals, the *results* are unequal in the same order.

45. If unequals are added to unequals of the same order, the *sums* are unequal in the same order.

46. Equal powers or equal roots of unequals produce *results* which are unequal in the same order.

47. If unequals are subtracted from equals, the *differences* are unequal in the reverse order.

48. A magnitude is greater than any of its parts.

49. If three quantities are so related that the first is greater than the second and the second is greater than the third, then the first is greater than the third.

50. Two quantities must possess one of three relations to each other; they are equal, or, the first is greater than the second, or, the first is less than the second.

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